Five-point two-loop master integrals in QCD

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work in collaboration with Thomas Gehrmann and Johannes Henn

Outline

Introduction

With LHC's RUN II higher precision of theoretical predictions is expected.

1-loop NLO established in the last decade as the new standard for high-multiplicity processes. BlackHat, Gosam, OpenLoops, NJet ...

2-loop NNLO is the current frontier
(although: N^3LO for inclusive Higgs production done [Anastasiou et al.]).

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- $2 \to 2$ processes calculated recently ($\gamma\gamma$, ZZ, $Z\gamma$, $W\gamma$, WW, $t\bar{t}$, Hj, Wj, jj)
  [Catani, Cieri, de Florian, Ferrera, Grazzini, Gehrmann, G.-De Ridder, Glover, Boughezal, Focke, Liu,Petriello, Czakon, Fiedler, Mitov, Kallweit, Maierhöfer, Rathlev, Chen, Jaquier, Giele, Melnikov, Caola, Schulze, Tejeda-Yeomans, Huss, Morgan . . .]

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- **$2 \to 3$ processes still open**

Introduction

Among NNLO bottle-necks:
two-loop scattering amplitudes $\longrightarrow$ **purely virtual contribution**.

At one-loop Feynman diagrams can be decomposed
into a small set of master integrals (MIs), all of which are known.

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Taking derivatives of the integr-als/-ands delivers a very powerful tool
- to reduce the amplitude to MIs,
- **to evaluate the integrals** (Differential Equation method).

Review of Differential-Equations method

Given a Feynman integral

$$G(a_1, a_2, \ldots, a_n) = \int \prod_{j=1}^{l} \frac{d^D k_j}{i \pi^{D/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} \quad , \quad \text{where } D_i = (k_j - p_i, - \ldots)^2$$

Integration by part identities

$$\int \prod_{j=1}^{l} \frac{d^D k_j}{i \pi^{D/2}} \left(\frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} \right) = 0$$

(v^μ is appropriately chosen vector, e.g. $k_j^\mu - p_1^\mu$)

$\rightarrow$ terms with same denominators D_i , but different indices $a_1, a_2, \ldots$

relate different integrals $\Longrightarrow$ we can reduce them to MIs.[Laporta alg.]

AIR[Anastasiou, Lazopoulos], Fire [Smirnov], Reduze [Studerus, Manteuffel], LiteRed [Lee]

Review of Differential-Equations method

Derivaties w.r.t external kinematic invariants, e.g. $s_{12} = (p_1 + p_2)^2$

$$\frac{\partial}{\partial s_{12}} \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} = \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{2\, s_{12}} \left((p_1+p_2)^\mu \frac{\partial}{\partial (p_1+p_2)^\mu} \right) \frac{1}{D_1^{a_1} \ldots D_n^{a_n}}$$

on the R.H.S. again: same D_i s , but different indices $a_1, a_2, \ldots$

reduced to master integrals using IBP relations

$\Rightarrow$ differential equations for MIs. [Gehrmann, Remiddi]

Codes used: Fire [Smirnov], Reduze [von Manteuffel]

Review of Differential-Equations method

MIs basis is not unique. Suitable choice considerably simplifies diff. eqs.:

$\partial_x \vec{f} = A(x,\varepsilon)\vec{f} \longrightarrow \partial_x \vec{f} = \varepsilon A(x)\vec{f}$ can be integrated order by order in ε.

[J. Henn]

$$\vec{f}(x,\varepsilon) = \vec{f}_0(x) + \varepsilon \vec{f}_1(x) + \varepsilon^2 \vec{f}_2(x) + \ldots \implies \begin{aligned} \vec{f}_0(x) &= \vec{f}_0 \\ \vec{f}_1(x) &= \int dx A(x)\,\vec{f}_0 \\ \vec{f}_2(x) &= \int dx A(x)\,\vec{f}_1(x) \\ &\ldots \end{aligned}$$

Solution (symbolic): $\vec{f}(\vec{x},\varepsilon) = P\exp\left[\varepsilon \int_\gamma dA\right] \vec{f}(\vec{x}_0,\varepsilon)$

Review of Differential-Equations method

Further simplification:

$$\partial_x \vec{f} = \varepsilon \sum_k \frac{A_k}{x - x_k} \vec{f} \quad \longrightarrow \quad d\vec{f}(\vec{x},\varepsilon) = \varepsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x},\varepsilon)$$

where the list of functions $\{\alpha_1, \dots \alpha_n\}$ is called **alphabet**.

Transcendental weight = number of successive integrations

Important consequence of canonical form $\partial_{\mathbf{x}} \mathbf{f} = \varepsilon\, \mathbf{A}(\mathbf{x})\, \mathbf{f}$

Starting from $\vec{f}_0(x) = \vec{f}_0$ $\rightarrow$ transcendentality-0 constant

$\Rightarrow$ each order in ε has uniform transcendentality .

Review of Differential-Equations method

Solutions expressed in terms of multiple polylogarithms
[Remiddi,Vermaseren; Gehrmann, Remiddi; Goncharov]

$$G(a_1,a_2,\ldots,a_n\,;x) = \int_0^x \frac{dt}{t-a_1} G(a_2,\ldots,a_n\,;t)\,,$$

$$\text{with } G(x) = 1\,, \quad G(0) = 0 \quad \text{and} \quad G(\vec{0}_n\,;x) = \frac{1}{n!}\log^n x\,.$$

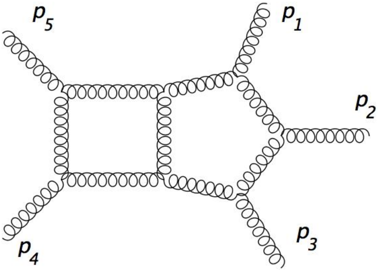
Simple example: $G(\vec{a}_n\,;x) = \frac{1}{n!}\log^n\left(1-\frac{x}{a}\right)$ with $\vec{a}_n = \{a,\ldots,a\}$

If $a_i \in \{1,-1,0\}$ $\longrightarrow$ Harmonic Polylogarithms.

2-loop five-point planar integrals

$$G_{\{a_1,\ldots,a_{11}\}} = \int \frac{d^D k_1\, d^D k_2}{(i\pi^{D/2})^2} \frac{D_9^{-a_9}\, D_{10}^{-a_{10}}\, D_{11}^{-a_{11}}}{D_1^{a_1}\, D_2^{a_2}\, D_3^{a_3}\, D_4^{a_4}\, D_5^{a_5}\, D_6^{a_6}\, D_7^{a_7}\, D_8^{a_8}}$$

$$\begin{aligned}
D_1 &= -k_1^2\,, \\
D_2 &= -(k_1+p_1)^2\,, \\
D_3 &= -(k_1+p_1+p_2)^2\,, \\
D_4 &= -(k_1+p_1+p_2+p_3)^2\,, \\
D_5 &= -k_2^2\,, \\
D_6 &= -(k_2+p_1+p_2+p_3)^2\,, \\
D_7 &= -(k_2+p_1+p_2+p_3+p_4)^2\,, \\
D_8 &= -(k_1-k_2)^2\,, \\
D_9 &= -(k_1+p_1+p_2+p_3+p_4)^2\,, \\
D_{10} &= -(k_2+p_1)^2\,, \\
D_{11} &= -(k_2+p_1+p_2)^2
\end{aligned}$$

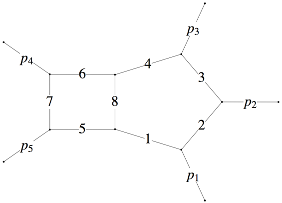


$$s_{ij} = (p_i+p_j)^2$$

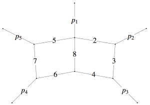
$$\vec{x} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\}$$

2-loop five-point planar integrals

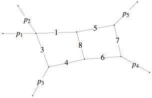
61 MIs



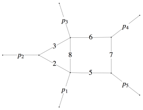
{46, G[1, {1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0}], 3}



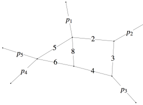
{45, G[1, {0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0}], 3}



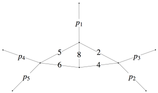
{44, G[1, {1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0}], 2}



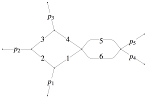
{40, G[1, {0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0}], 2}



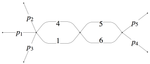
{38, G[1, {0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 0}], 1}



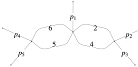
{28, G[1, {0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0}], 1}



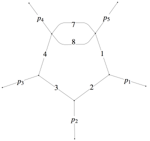
{33, G[1, {1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0}], 1}



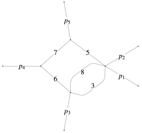
{9, G[1, {1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0}], 1}



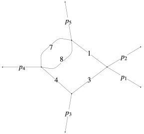
{11, G[1, {0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0}], 1}



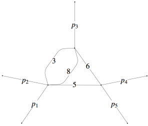
{34, G[1, {1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0}], 2}



{32, G[1, {0, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0}], 1}



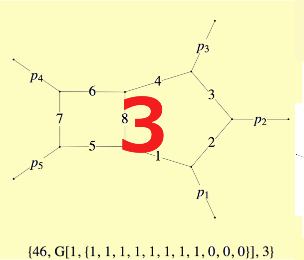
{20, G[1, {1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0}], 1}

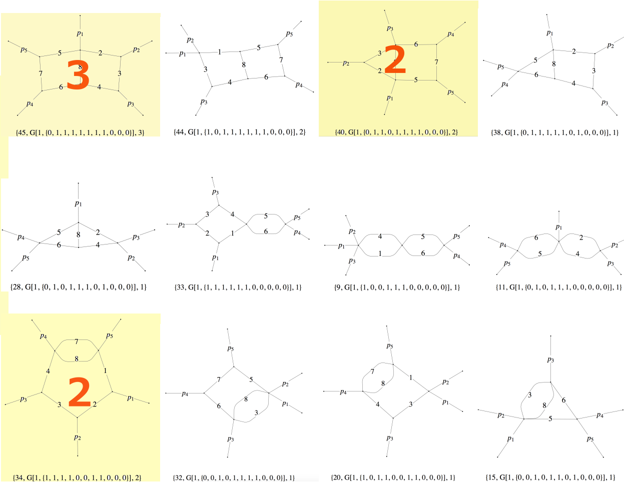


{15, G[1, {0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0}], 1}

2-loop five-point planar integrals

61 MIs , 10 new





≤ 4 point MIs known

[Gehrmann, Remiddi]

2-loop five-point planar integrals

Alphabet of 24 letter

$$\{ \; s_{12} \; , \; s_{12}-s_{34} \; , \; s_{12}+s_{23} \; , \; s_{12}-s_{34}-s_{45} \; , \; \ldots \; ,$$
$$(s_{23}-s_{51})\sqrt{\Delta} + s_{12}\,s_{23}^2 - s_{34}\,s_{23}^2 + s_{34}\,s_{45}\,s_{23} - 2\,s_{12}\,s_{51}\,s_{23}$$
$$+ s_{34}\,s_{51}\,s_{23} + s_{45}\,s_{45}\,s_{23} + s_{12}\,s_{51}^2 - s_{45}\,s_{51}^2 + s_{34}\,s_{45}\,s_{23} \; \}$$

Δ is the Gram determinant.
With a suitably chosen parametrization, $\Delta \to$ perfect square

$$\begin{aligned}
s_{12} &= z_1 \, , \\
s_{23} &= z_1\, z_2\, z_4 \, , \\
s_{34} &= (z_1/z_2)\left[z_3\,(z_4-1) + z_2\,z_4 + z_2\,z_3\,(z_4-z_5)\right] , \\
s_{45} &= z_1\, z_2\,(z_4-z_5) \, , \\
s_{51} &= z_1\, z_3\,(1-z_5)
\end{aligned}$$

obtained by using Momentum Twistor variables [Hodges 0905.1473]

Boundary conditions

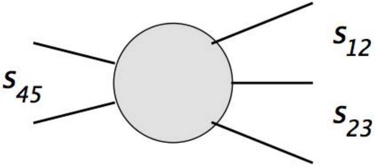
Boundary values can be obtained from physical conditions,
in kinematic limits with singular diff. eq. but regular integrals.

No singularities in the Euclidean region $s_{i,i+1} < 0$.

Un-physical singularities
appear in the limit

$$s_{45} \to s_{12} + s_{23}$$

and they need to cancel.



$\longrightarrow$ no need to compute any additional integrals.

Boundary conditions

$\Delta = 0$ defines hypersurface where divergencies need to cancel.

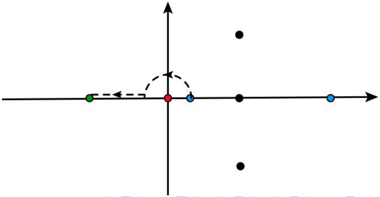
The symmetric point $\vec{x}_{sym} = \{-1,-1,-1,-1,-1\}$

is connected to the $\Delta = 0$ surface by

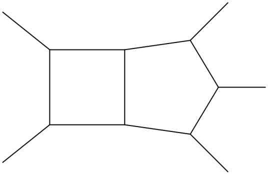
$$\vec{f}(\vec{x},\varepsilon) = P\exp\left[\varepsilon \int_\gamma dA\right] \vec{f}(\vec{x}_0,\varepsilon)$$

path $\gamma = \{-\frac{y}{(1-y)^2},-1,-1,-1,-1\} \longrightarrow$ reduced alphabet .

$$\text{Sym. pt} \quad \rightarrow \quad y = \frac{3 \pm \sqrt{5}}{2}$$
$$\Delta = 0 \quad \rightarrow \quad y = -1$$



Canonical basis and boundary conditions

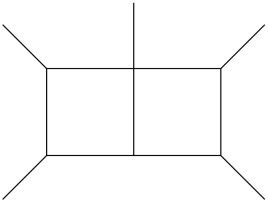


$$f_{59} = \epsilon^2 \frac{s_{12}s_{23}s_{45}s_{35}}{\sqrt{\Delta}} G[\{1,1,1,1,1,1,1,1,-1,-1,0\}]$$
$$+\epsilon^2 s_{12}s_{23}s_{45} \frac{\mathrm{tr}[\not{p}_1\not{p}_2\not{p}_3\not{p}_4]}{4\sqrt{\Delta}} G[\{1,1,1,1,1,1,1,1,-1,0,0\}]$$
$$\to \epsilon^3 c_{5,3} + \ldots$$

$$f_{60} = \epsilon^2 s_{12}s_{23}s_{45} G[\{1,1,1,1,1,1,1,1,-1,0,0\}]$$
$$\to -3 - \frac{11}{6}\pi^2\epsilon^2 + \ldots$$

$$f_{61} = \epsilon^2 \frac{s_{12}s_{23}s_{45}s_{35}}{\sqrt{\Delta}} \Big[s_{45} G[\{1,1,1,1,1,1,1,1,0,-1,0\}]$$
$$-G[\{1,1,1,1,1,1,1,1,-1,-1,0\}] \Big]$$
$$-\epsilon^2 s_{12}s_{23}s_{45} \frac{\mathrm{tr}[\not{p}_1\not{p}_2\not{p}_3\not{p}_4]}{4\sqrt{\Delta}} G[\{1,1,1,1,1,1,1,1,-1,0,0\}]$$
$$\to -\epsilon^3 c_{5,3} + \ldots$$

Canonical basis and boundary conditions

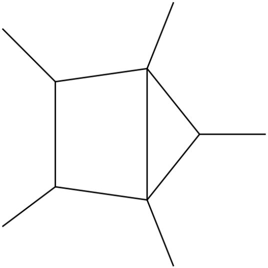


$$f_{56} = \epsilon^2 s_{12} s_{45} s_{51} G[\{1,1,1,0,1,1,1,1,0,0,0\}]$$
$$\to -\frac{9}{4} + \frac{29}{24}\pi^2 \epsilon^2 + \dots$$

$$f_{57} = \epsilon^2 s_{12} s_{45} G[\{1,1,1,0,1,1,1,1,-1,0,0\}]$$
$$\to -\frac{3}{2} - \frac{3}{4}\pi^2 \epsilon^2 + \dots$$

$$f_{58} = \epsilon^2 \frac{s_{12} s_{45} s_{24}}{\sqrt{\Delta}} G[\{1,1,1,0,1,1,1,1,-1,-1,0\}] + \dots$$
$$-s_{12} s_{45} \frac{\mathrm{tr}[\not p_1 \not p_2 \not p_3 \not p_4]}{4\sqrt{\Delta}} \Big[s_{51} G[\{1,1,1,0,1,1,1,1,0,0,0\}]$$
$$+G[\{1,1,1,0,1,1,1,1,-1,0,0\}] \Big] \dots$$
$$\to -\epsilon^3 c_{5,3} + \dots$$

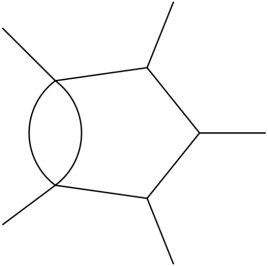
Canonical basis and boundary conditions



$$f_{46} = \varepsilon^2 s_{45}(s_{34} - s_{51}) G[\{0,1,1,0,1,1,1,1,0,0,0\}]$$
$$\to 0 + \ldots$$

$$f_{47} = \varepsilon \frac{s_{12}s_{23}s_{34}s_{45}s_{51}}{\sqrt{\Delta}} G[\{0,1,1,0,1,1,1,1,0,0,0\}] +$$
$$+\varepsilon^2 \frac{\text{Num}}{\sqrt{\Delta}} G[\{0,1,1,0,1,1,1,2,0,0,0\}] + \ldots$$
$$\to \varepsilon^3 c_{5,3} + \ldots$$

Canonical basis and boundary conditions



$$f_{37} = -\varepsilon s_{12} s_{23} \frac{\mathrm{tr}[\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4]}{2\sqrt{\Delta}} G[\{1,1,1,1,0,0,2,1,-1,0,0\}] + \varepsilon \frac{s_{12} s_{23} s_{34} s_{45} s_{51}}{\sqrt{\Delta}} G[\{1,1,1,1,0,0,2,1,0,0,0\}]$$

$$\rightarrow \frac{3}{2} c_{5,3} + \ldots$$

$$f_{38} = \varepsilon s_{12} s_{23} G[\{1,1,1,1,0,0,2,1,-1,0,0\}]$$

$$\rightarrow -2 + \varepsilon^2 \pi^2 + \varepsilon^3 \frac{55}{3} \zeta_3 + \ldots$$

Applications: All-plus amplitude

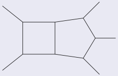
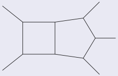
We have applied our integrals to the **all-plus amplitude** (leading-colour).
[Badger, Frellesvig, Zhang]

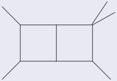
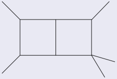
$$\mathcal{A}_5(1^+,2^+,3^+,4^+,5^+)|_{\text{leading colour}} =$$

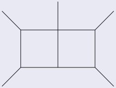
$$g_s^7 N_c^2 c_\Gamma^2 \sum_{\sigma\in S_5} \text{tr}(T^{a_{\sigma(1)}}\, T^{a_{\sigma(2)}}\, T^{a_{\sigma(3)}}\, T^{a_{\sigma(4)}}\, T^{a_{\sigma(5)}})\;\; A_5^{(2)}(\sigma(1)^+,\sigma(2)^+,\sigma(3)^+,\sigma(4)^+,\sigma(5)^+)$$

$$A_5^{(2)} = A_5^{(2)\,\text{bare}} - \frac{11}{3\varepsilon} A_5^{(1)} \qquad\qquad A_5^{(2)\,\text{bare}} = \sum_{i=1}^{5} A_5^{[P]}(1^+,2^+,3^+,4^+,5^+)$$

$$A_5^{[P]}(1^+,2^+,3^+,4^+,5^+) = \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \Bigg\{$$

c_{431} $\times$  $[F_1]$ $+$ c^T_{431} $\times$  $[F_1\,(k_1+p_5)^2]$

$+$ $c_{331;M_1}$ $\times$  $[F_1]$ $+$ $c_{331;M_2}$ $\times$  $[F_1]$

$+$ $c_{331;L}$ $\times$  $[F_1]$ $+$ butterflies$[F_3]$ $\Bigg\}$,

$F_1\,, F_3 \rightarrow \left(k_1^{[-2\epsilon]}\right)^2 \,,\, \left(k_2^{[-2\epsilon]}\right)^2$ and $2\,k_1^{[-2\epsilon]}\cdot k_2^{[-2\epsilon]}$ polynomials in numerator

$\int \frac{d^{-2\epsilon}k_1^{[-2\epsilon]}}{(2\pi)^{2\epsilon}} \int \frac{d^{-2\epsilon}k_2^{[-2\epsilon]}}{(2\pi)^{2\epsilon}}$ translates into $I_A[F_1] \longrightarrow 4\,\epsilon\,(\epsilon-1) \sum_{i,j\in P(A)} \mathbf{i}^+\mathbf{j}^+\, I_A^{[8-2\epsilon]}$

[Bern, De Freitas, Dixon ; Badger, Frellesvig, Zhang]

One-loop amplitude

At one loop we have [Bern, Dixon, Dunbar, Kosower]

$$A^{(1)}(1^+2^+3^+4^+5^+) = \frac{-i\varepsilon(1-\varepsilon)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\Bigg(2\,(2-\varepsilon)\,\mathrm{tr}_5\, I^{[10-2\varepsilon]}_{[5;12345]}[1]$$

$$+\, s_{12}s_{23}\, I^{[8-2\varepsilon]}_{4;1234}[1] \,+\, s_{23}s_{34}\, I^{[8-2\varepsilon]}_{4;2345}[1] \,+\, s_{34}s_{45}\, I^{[8-2\varepsilon]}_{4;3451}[1] \,+\, s_{45}s_{51}\, I^{[8-2\varepsilon]}_{4;5123}[1]\Bigg)$$

Leading power of ε $\rightarrow \varepsilon^0$ contribution is

$$A^{(1)}(1^+2^+3^+4^+5^+) = \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left(-\frac{1}{6}F_5^{(1)}\right),$$

with

$$F_5^{(1)} = v_1v_2 + v_2v_3 + v_3v_4 + v_4v_5 + v_5v_1 + \mathrm{tr}_5\,.$$

$$\mathrm{tr}_5 = \mathrm{tr}[\gamma_5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4] = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]$$

Infrared and UV structure

The infrared and ultraviolet structure is described by the one-loop amplitude [Catani]

$$A^{(2)}(1^+2^+3^+4^+5^+) = A^{(1)}(1^+2^+3^+4^+5^+)\left[-\sum_{i=1}^{5}\frac{1}{\varepsilon^2}\left(\frac{\mu^2}{-v_i}\right)^{\varepsilon}-\frac{11}{3}\frac{1}{\varepsilon}\right]$$

$$+\frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left(-\frac{1}{6}F_5^{(2)}\right)+O(\varepsilon),$$

$F_5^{(2)}$ is the finite remainder function at two loops.

Noteworthy : weight **1**, **3** and **4** functions only originates from 1-loop ampl.

$\Rightarrow$ will not be present in $F_5^{(2)}$

Parity odd and parity even

It is convenient to split the amplitude into two contribution

$$\begin{aligned} \mathrm{tr}_5 &= \mathrm{tr}[\gamma_5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4] = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41] \\ &= 4\,i\,\varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma \end{aligned}$$

$$\mathrm{tr}_5^2 = \mathrm{tr}[\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4]^2 - 4\, s_{12}\, s_{23}\, s_{34}\, s_{41} = \Delta$$

Parity even part depends only on Mandelstam invariants.

(at one loop: eight-dimensional boxes)

$\longrightarrow$ yields terms with trascendental weight up to weight 3.

Parity odd part proportional to tr_5 .

(at one loop: ten-dimensional pentagon)

$\longrightarrow$ yields terms with trascendental weight up to weight 4.

Results

$$A^{(2)}_{5\,\text{plus}} = A^{(1)}_{5\,\text{plus}}\left[-\sum_{i=1}^{5}\frac{1}{\varepsilon^2}\left(\frac{\mu^2}{-v_i}\right)^{\varepsilon} - \frac{11}{3}\frac{1}{\varepsilon}\right] + \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left(-\frac{1}{6}F^{(2)}_5\right) + O(\varepsilon)\,,$$

We find: $v_1 = s_{12}\,,\ v_1 = s_{12}\,,\ v_2 = s_{23}\,,\ v_3 = s_{34}\,,\ v_4 = s_{45}\,,\ v_5 = s_{51}$

$$F^{(2)}_5 = \frac{5}{2}\zeta_2 F^{(1)}_5 + \sum_{i=1}^{5}\sigma^i\left[\frac{v_5(v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 - \mathrm{tr}_5)}{(v_2+v_3-v_5)}F_{23,5}\right]$$
$$+ \frac{1}{3}\sum_{i=1}^{5}\sigma^i\left[\frac{1}{2}\frac{(v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 - \mathrm{tr}_5)^2}{v_1v_4} + 10v_1v_2 + 2v_1v_3\right].$$

with

$$F_{23,5} = \frac{1}{2}\mathrm{Li}_2\left(1-\frac{v_5}{v_2}\right) - \frac{1}{2}\mathrm{Li}_2\left(1-\frac{v_2}{v_5}\right) + \frac{1}{2}\mathrm{Li}_2\left(1-\frac{v_5}{v_3}\right) - \frac{1}{2}\mathrm{Li}_2\left(1-\frac{v_3}{v_5}\right)$$
$$+ \frac{1}{4}\log^2\frac{v_2}{v_5} + \frac{1}{4}\log^2\frac{v_3}{v_5} - \log\frac{v_2}{v_5}\log\frac{v_3}{v_5} + \zeta_2\,.$$

Results

$$A^{(2)}_{5\,\mathrm{plus}} = A^{(1)}_{5\,\mathrm{plus}} \left[-\sum_{i=1}^{5} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{-v_i} \right)^{\varepsilon} - \frac{11}{3} \frac{1}{\varepsilon} \right] + \frac{i}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \left(-\frac{1}{6} F^{(2)}_5 \right) + O(\varepsilon) \,,$$

We find: $v_1 = s_{12}\,,\ v_1 = s_{12}\,,\ v_2 = s_{23}\,,\ v_3 = s_{34}\,,\ v_4 = s_{45}\,,\ v_5 = s_{51}$

$$F^{(2)}_5 = \frac{5}{2} \zeta_2 F^{(1)}_5 + \sum_{i=1}^{5} \sigma^i \left[\frac{v_5 (v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 - \mathrm{tr}_5)}{(v_2 + v_3 - v_5)} F_{23,5} \right]$$
$$+ \frac{1}{3} \sum_{i=1}^{5} \sigma^i \left[\frac{1}{2} \frac{(v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 - \mathrm{tr}_5)^2}{v_1 v_4} + 10 v_1 v_2 + 2 v_1 v_3 \right] .$$

$\rightarrow$ one-loop two-mass easy box function in six dimensions.
It can also be written as

$$F_{23,5} = \zeta_2 - \mathrm{Li}_2 \left(\frac{v_5 - v_3}{v_2} \right) - \mathrm{Li}_2 \left(\frac{v_5 - v_2}{v_3} \right) + \mathrm{Li}_2 \left(\frac{(v_5 - v_2)(v_5 - v_3)}{v_2 v_3} \right)$$

Checks

Checked against numerical results of [Badger, Frellesvig, Zhang]

Double and single pole cancellation provides non-trivial check.

Factorization properties of scattering amplitudes in the soft and **collinear limit** allow to connect provides a way to check them.

In the collinear limit $p_4||p_5$, we expect to find

$$\begin{aligned} A_5^{(2)}(1^+,2^+,3^+,4^+,5^+) \rightarrow & \text{Split}^{P\rightarrow 45\,(1)}(P^-,4^+,5^+)A_4^{(1)}(1^+,2^+,3^+,P^+) \\ & + \text{Split}^{P\rightarrow 45\,(1)}(P^+,4^+,5^+)A_4^{(1)}(1^+,2^+,3^+,P^-) \\ & + \text{Split}^{P\rightarrow 45\,(0)}(P^-,4^+,5^+)A_4^{(2)}(1^+,2^+,3^+,P^+)\,. \end{aligned}$$

Check not completed yet, but looking promising: weight-2 part in progress.

Summary and Outlook

- I have presented the computation of five-point two-loop MIs (planar).
- Results obtained using the Differential-Equation method, with MIs basis that makes the diff. eq. system canonical.
- Boundary conditions obtained by requiring the cancellation of spurious singularities in diff. eqs. $\rightarrow$ No further integration required.
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- Analytic continuation outside Euclidean region ($\rightarrow$ physical region).
- Non-planar integrals: in progress.