Theoretical Particle Physics Seminar — University of Zurich (September 28, 2020)

Factorization at Subleading Power and Endpoint Divergences in Soft-Collinear Effective Theory

Matthias Neubert

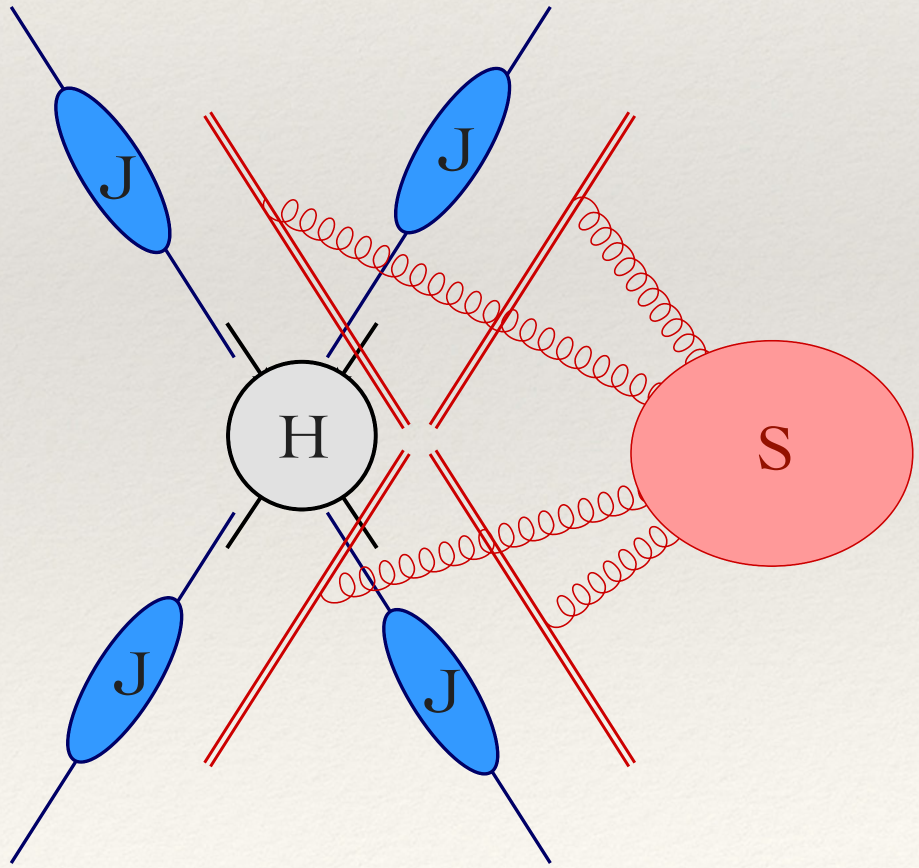
PRISMA+ Cluster of Excellence & MITP
Johannes Gutenberg University Mainz

Introduction

- Factorization of scales is a fundamental concept in HEP:
  - LHC cross section ~ $\sigma_{\text{parton}} \otimes$ PDFs
  - basis for the resummation of large logarithmic corrections

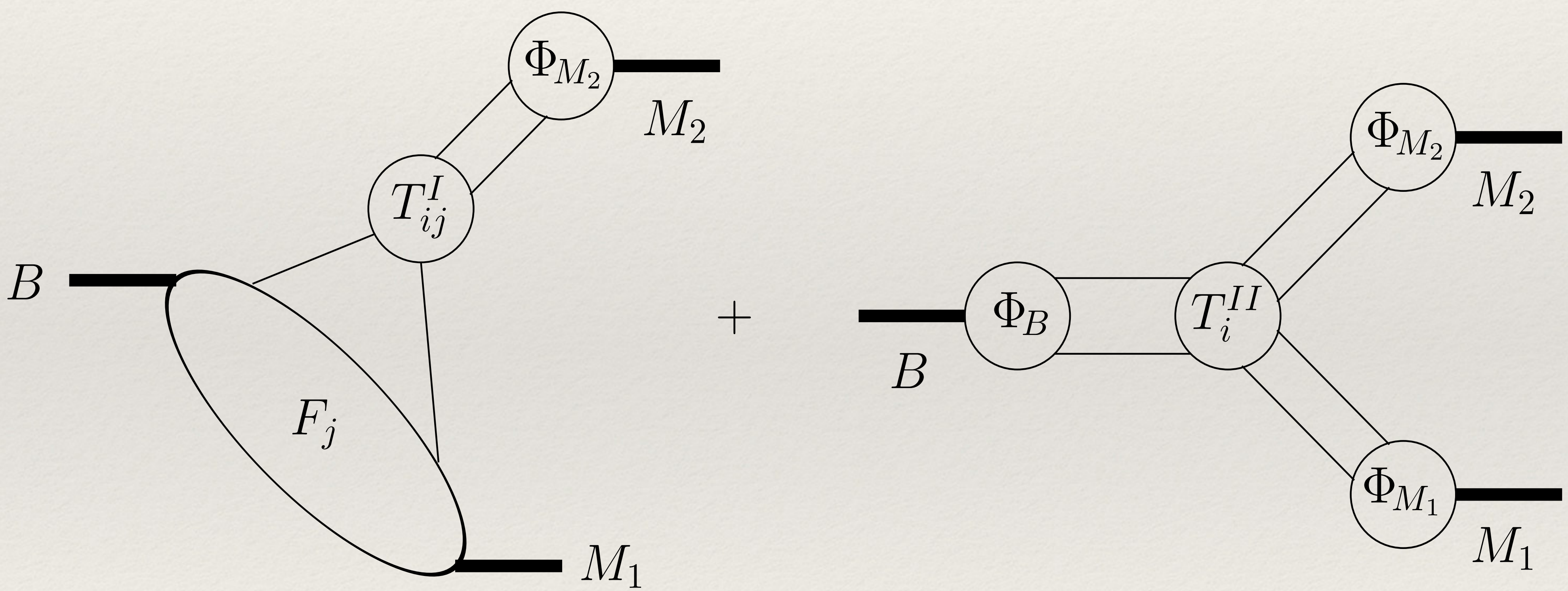
- **Soft-collinear effective theory** (SCET) provides a framework for studying factorization and resummation for processes involving light energetic particles using tools of effective field theory (EFT)

  [Bauer et al. 2000, 2001; Beneke et al. 2002]



Introduction

- SCET was constructed as the EFT for **QCD factorization**, as applied to hadronic decays of B mesons:



[Beneke, Buchalla, MN, Sachrajda 1999, 2000]

Introduction

- Puzzling observation:
  - some power-suppressed corrections (e.g. weak annihilation) involve divergent convolution integrals $\sim \int_0^1 \frac{dx}{x}$, which were interpreted as a **breakdown of factorization** [Beneke, Buchalla, MN, Sachrajda 2000]
  - proof of factorization requires showing that certain soft-collinear modes parameterizing these endpoint contributions cancel out to all orders [Hill, MN 2003; Becher, Hill, MN 2005]
- It was believed that SCET may overcome the problem of endpoint divergences ("everything factorizes") or at least shed new light on it

Introduction

- Conventional EFTs provide a systematic expansion in inverse powers of a large scale Q:

$$\mathcal{L}_{\text{eff}} = \sum_i C_i(Q,\mu)\, O_i(\mu) + \frac{1}{Q}\sum_j C_j^{(1)}(Q,\mu)\, O_j^{(1)}(\mu) + \frac{1}{Q^2}\sum_k C_k^{(2)}(Q,\mu)\, O_k^{(2)}(\mu) + \dots$$

- Examples: $\mathcal{H}_{\text{eff}}^{\text{weak}}$, χPT, HQET, NRQCD, SMEFT, …
- Extension to higher orders "straightforward if tedious"
  - χPT: 2, 12, 117, 1959, 45171, 1170086, … [Graf et al. 2020]
  - SMEFT: 12, 3045, 1542, 44807, 90456, 2092441, … [Henning, Lu, Melia, Murayama 2015]

Introduction

- SCET is more complicated in several aspects:
  - operators contain **non-local products of fields** (unavoidable consequence of $E \sim Q$ but $p^2 \ll Q^2$), need to introduce **Wilson lines** for gauge invariance
  - Wilson coefficients depend on large momentum components in addition to heavy masses of particles integrated out
  - fields are split up in **momentum modes** (method of regions):

[Beneke, Smirnov 1997]

$$\phi(x) \to \underbrace{\phi_{n_1}(x) + \phi_{n_2}(x) + \dots}_{\text{collinear}} + \underbrace{\phi_s(x) + \phi_{us}(x) + \dots}_{\text{soft}}$$

collinear
(regions of large momentum flow)

soft

Introduction

- SCET is more complicated in several aspects:
  - hard modes are integrated out
    (Wilson coefficients = hard matching coefficients)
  - different collinear sectors appear decoupled in the effective Lagrangian except for soft interactions
  - soft interactions can be decoupled by means of field redefinitions
    $\to$ **factorization theorems**
  - large logarithms can be resummed systematically by solving RGEs

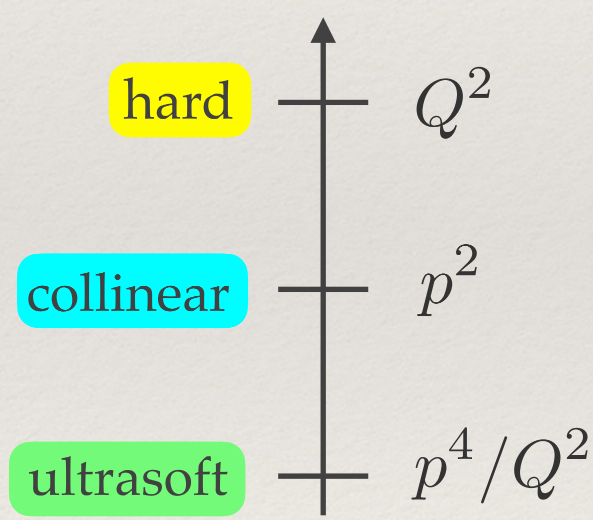
Introduction

- Typical SCET factorization theorem:

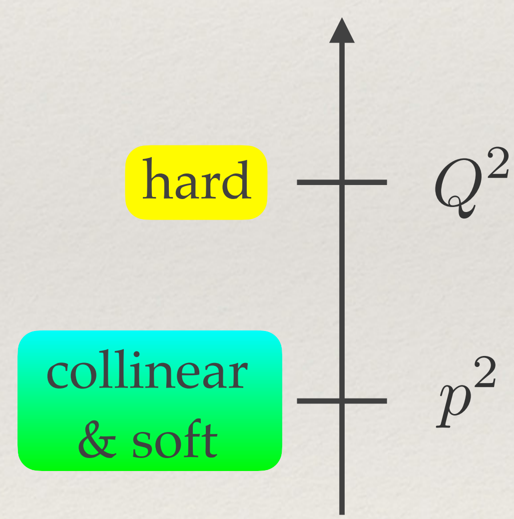
$$\sigma \sim H \int J \otimes J \otimes S$$

hard — collinear — soft

- Two common scale hierarchies:



SCET-1



SCET-2

In SCET-2 the product $J \otimes J \otimes S$ contains an extra dependence on Q^2 due to the **collinear anomaly**.

[Becher, MN 2010]

Introduction

- Examples:
  - threshold resummation for DIS, DY, Higgs $t\bar{t}$ production, …
  - p_T resummation, jet vetoes, event shapes, jet substructure, …
  - non-global logarithms, super-leading logarithms (ongoing work)
  - high-order structure of IR divergences of scattering amplitudes, subtractions methods for N^nLO fixed-order calculations (e.g. based on N-jettiness)

[Becher, MN et al. 2006-2016; …]

Introduction

- Extension to next-to-leading power?
  - generically (all known examples), find **endpoint-divergent convolution integrals**! [Beneke et al., Moult et al., Stewart et al., MN et al. 2018-2020; …]
  - upsets scale separation and breaks factorization
  - prevents systematic resummation of large logarithms
  - failure of standard OPE based on dimensional regularization and $\overline{\text{MS}}$ subtractions
- Questions usefulness of entire SCET framework!
  - a hard problem; many groups world-wide work on this…

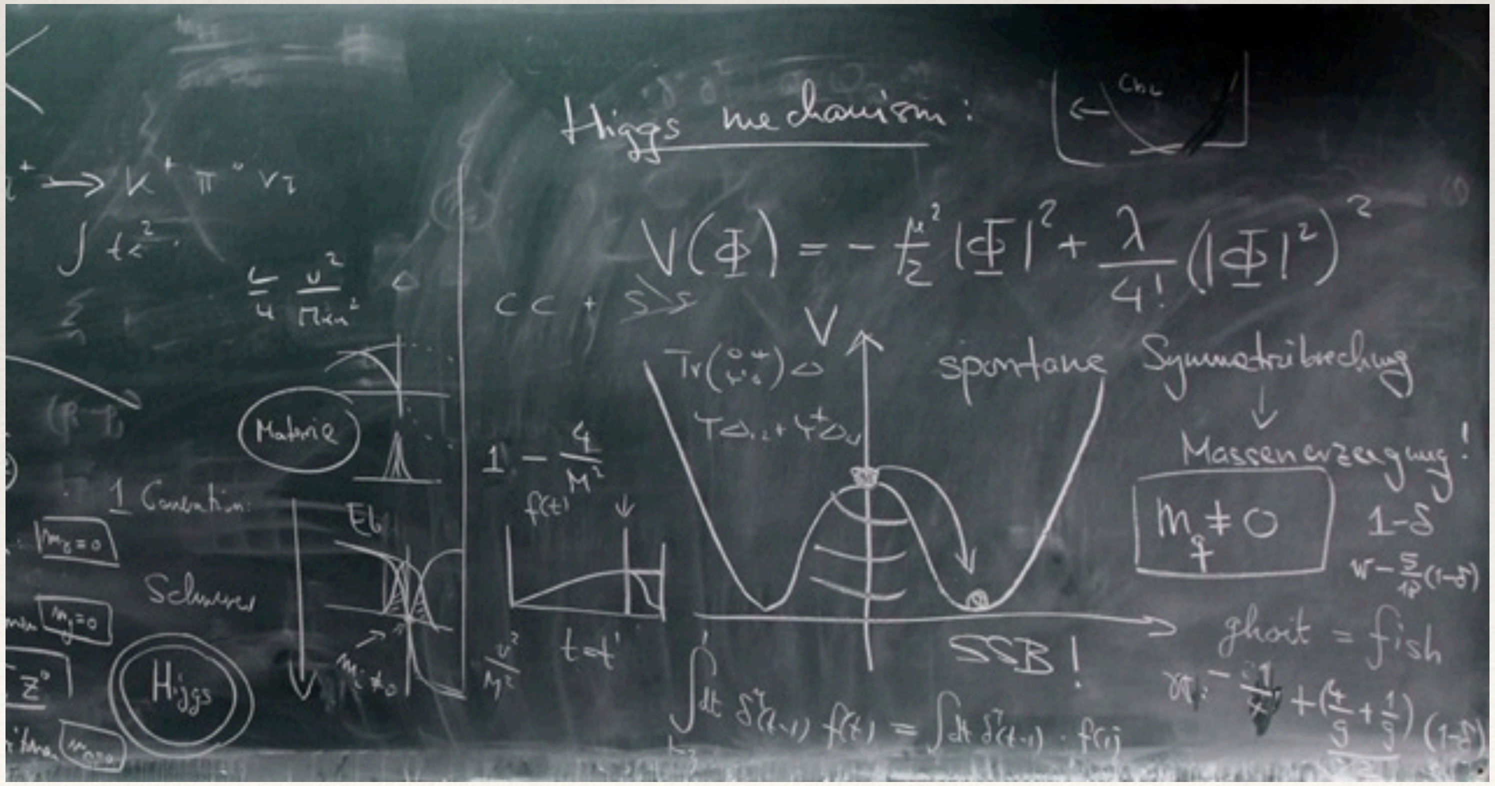
Organizers:

http://scet.itp.unibe.ch/

Thomas Becher, Christoph Greub, Thomas Rauh, Xiaofeng Xu, Marcel Balsiger, Samuel Favrod, Francesco Saturnino

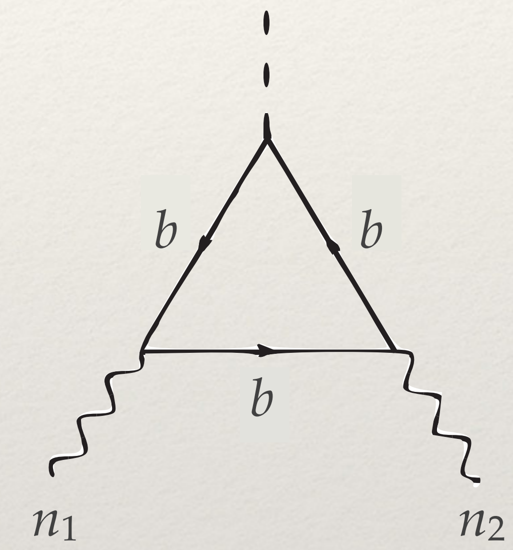
First SCET factorization theorem at subleading power

Liu, MN: 1912.08818 (JHEP)
Liu, Mecaj, MN, Yang: 2009.04456 & 2009.06779
Liu, MN: 2003.03393 (JHEP)
Liu, Mecaj, MN, Yang, Fleming: 2005.03013 (JHEP)



A subleading-power observable

- Consider b-quark induced contribution to $h \to \gamma\gamma$ decay amplitude (pseudo observable)
  - this and related $gg \to h$ process may be relevant for high-precision Higgs studies, but here are considered for academic purposes mainly
  - "sufficiently complicated but simple enough"

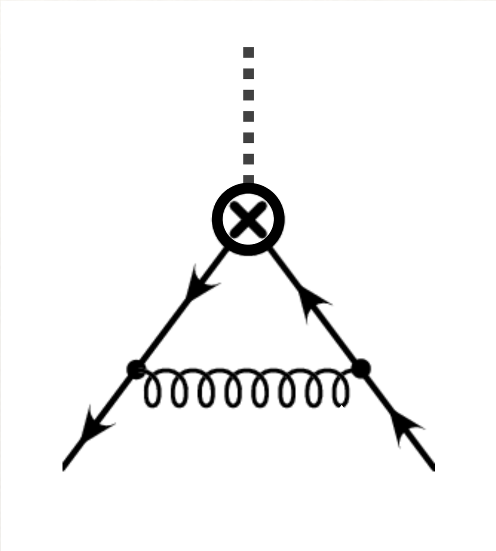


- Relevant modes are hard, collinear (n_1 and n_2) and soft, with SCET-2 scaling
- Scale hierarchy: $m_b^2 \ll M_h^2$

A subleading-power observable

- Same momentum regions appear in analysis of the Sudakov form factor (e.g. electroweak Sudakov resummation)
  - standard factorization theorem without endpoint divergences:

$$\sigma \sim H \int J \otimes J \otimes S$$



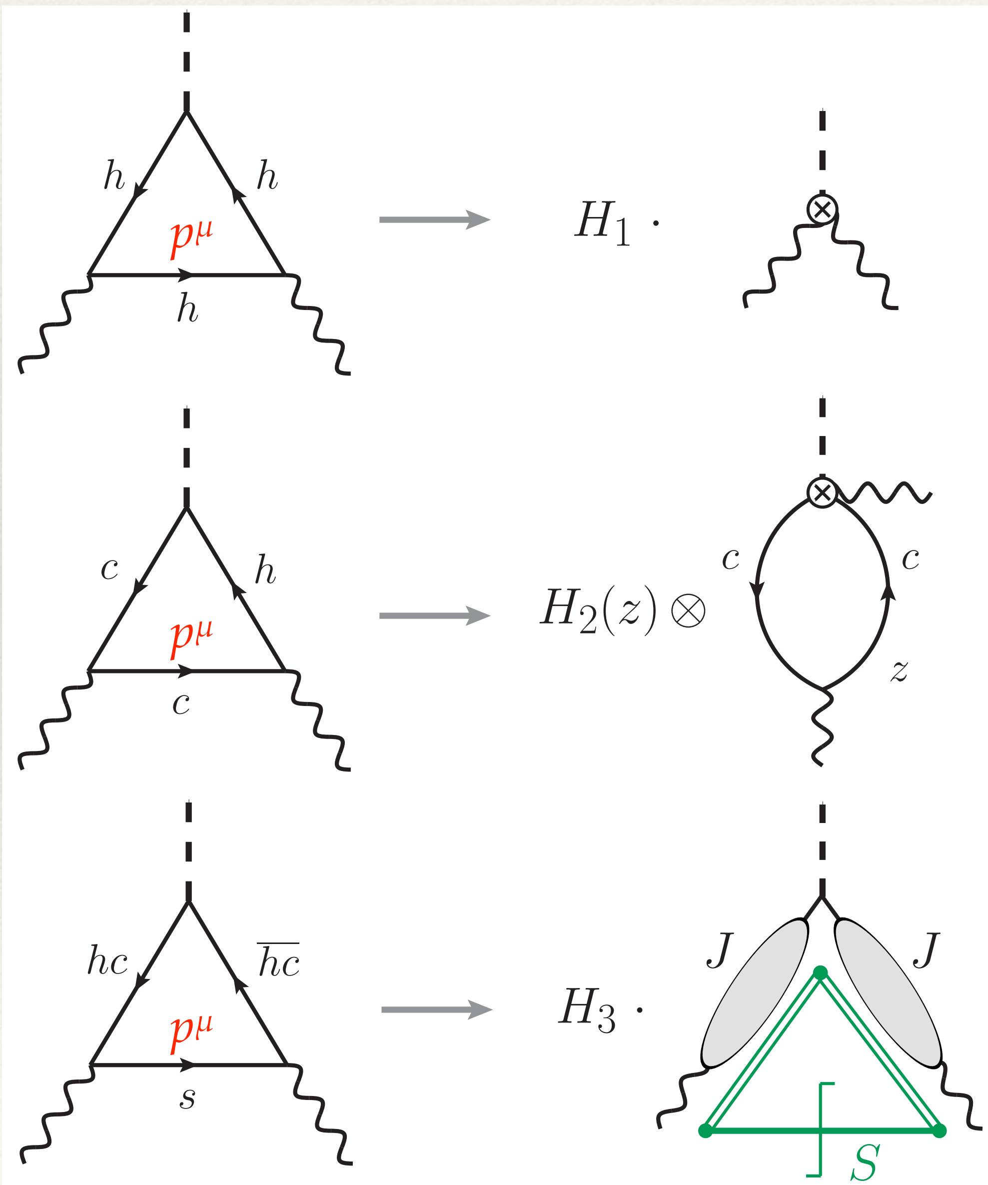
  - a single, leading-order SCET operator arises at $O(\lambda^2)$:

$$V_\mu \, \underbrace{\bar{\mathcal{X}}_{n_1}}_{\lambda} \gamma_\perp^\mu \underbrace{\mathcal{X}_{n_2}}_{\lambda}$$

  - crucial difference: **soft quark** can appear at subleading power

A subleading-power observable

- Relevant momentum regions at 1-loop order:



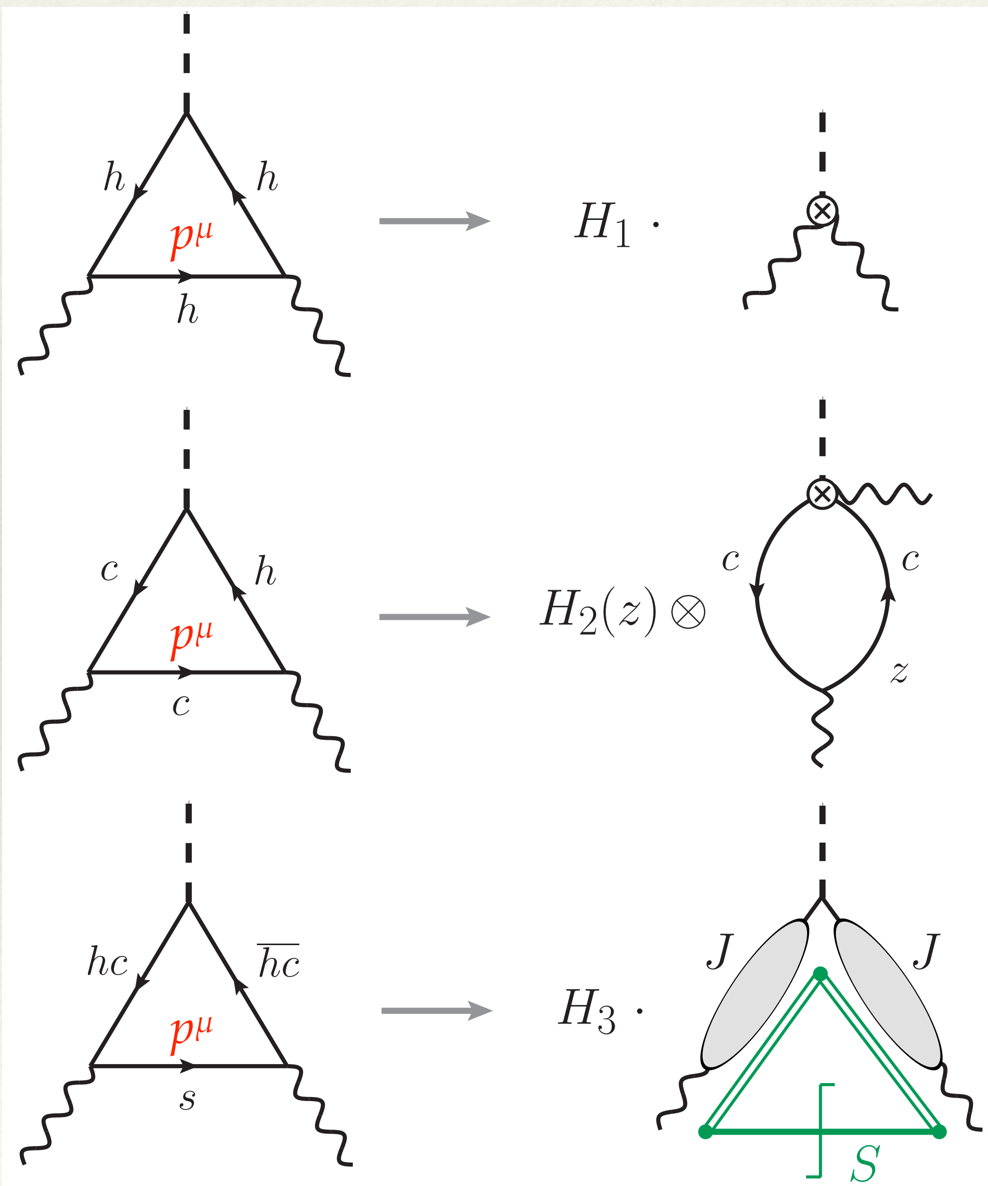
$p^\mu \sim M_h$ hard

$p^\mu \parallel k_1\,, \quad p^2 \sim m_b^2$ n_1-collinear

$p^\mu \sim m_b$ soft

A subleading-power observable

- Relevant momentum regions at 1-loop order: $\lambda \sim m_b/M_h$



$p^\mu \sim M_h\,(1, 1, 1)$ hard

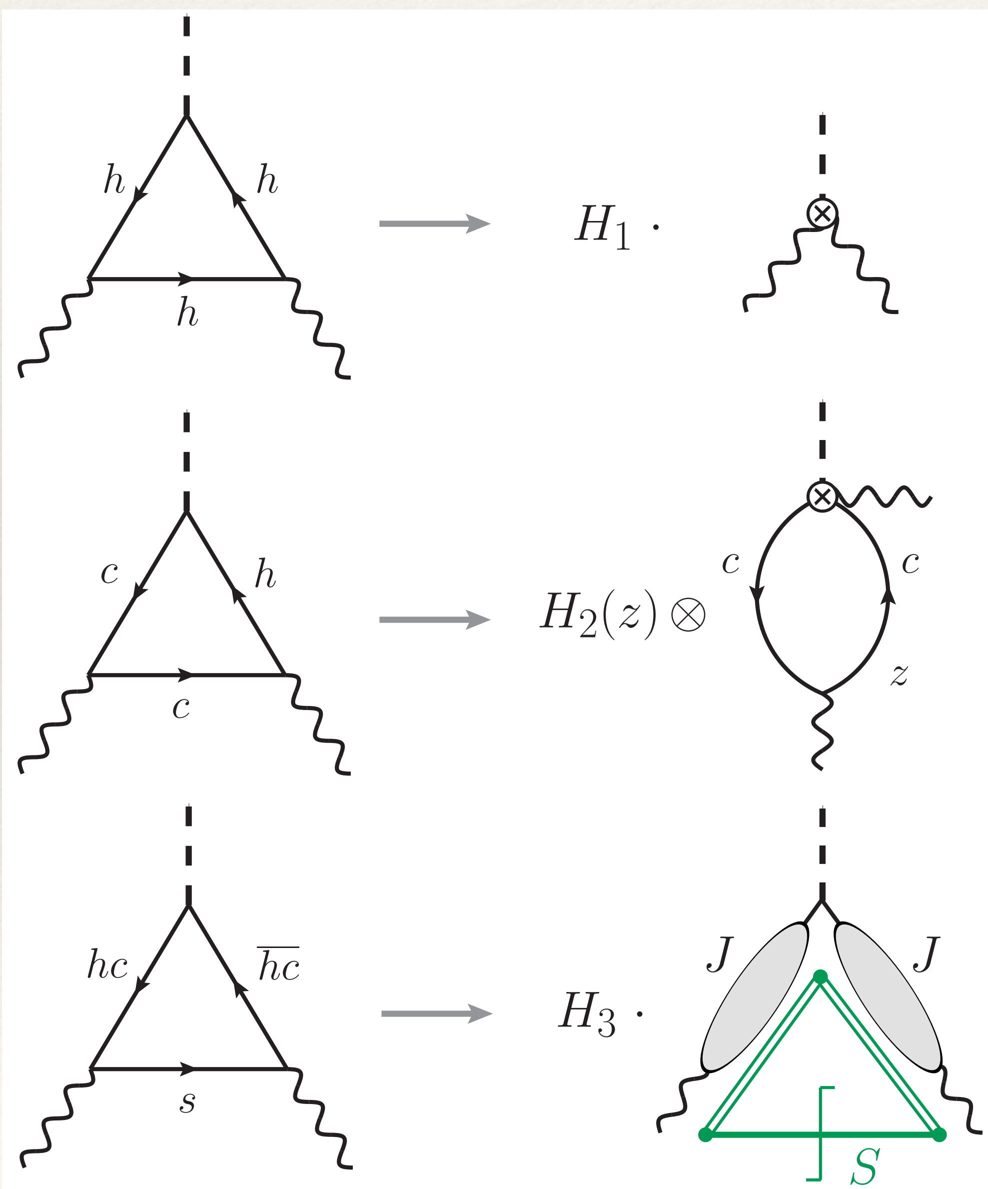
$n_1 \; n_2 \; \perp$

$p^\mu \sim M_h\,(1, \lambda^2, \lambda)$ n_1-collinear

$p^\mu \sim M_h\,(\lambda, \lambda, \lambda)$ soft

A subleading-power observable

- Corresponding SCET operators at $O(\lambda^3)$: $\lambda \sim m_b/M_h$



$$O_1^{(0)} = \underset{\lambda}{\frac{m_b}{e_b^2}}\, h\, \underset{\lambda}{\mathcal{A}_{n_1}^{\perp\mu}}\, \underset{\lambda}{\mathcal{A}_{n_2,\mu}^{\perp}}$$

dressed collinear photon fields

$$O_2^{(0)}(z) = h \left[\underset{\lambda}{\bar{\mathcal{X}}_{n_1}} \gamma_\perp^\mu \frac{\not{\bar{n}}_1}{2}\, \delta(z\bar{n}_1\cdot k_1 + i\bar{n}_1\cdot\partial)\, \underset{\lambda}{\mathcal{X}_{n_1}} \right] \underset{\lambda}{\mathcal{A}_{n_2,\mu}^{\perp}}$$

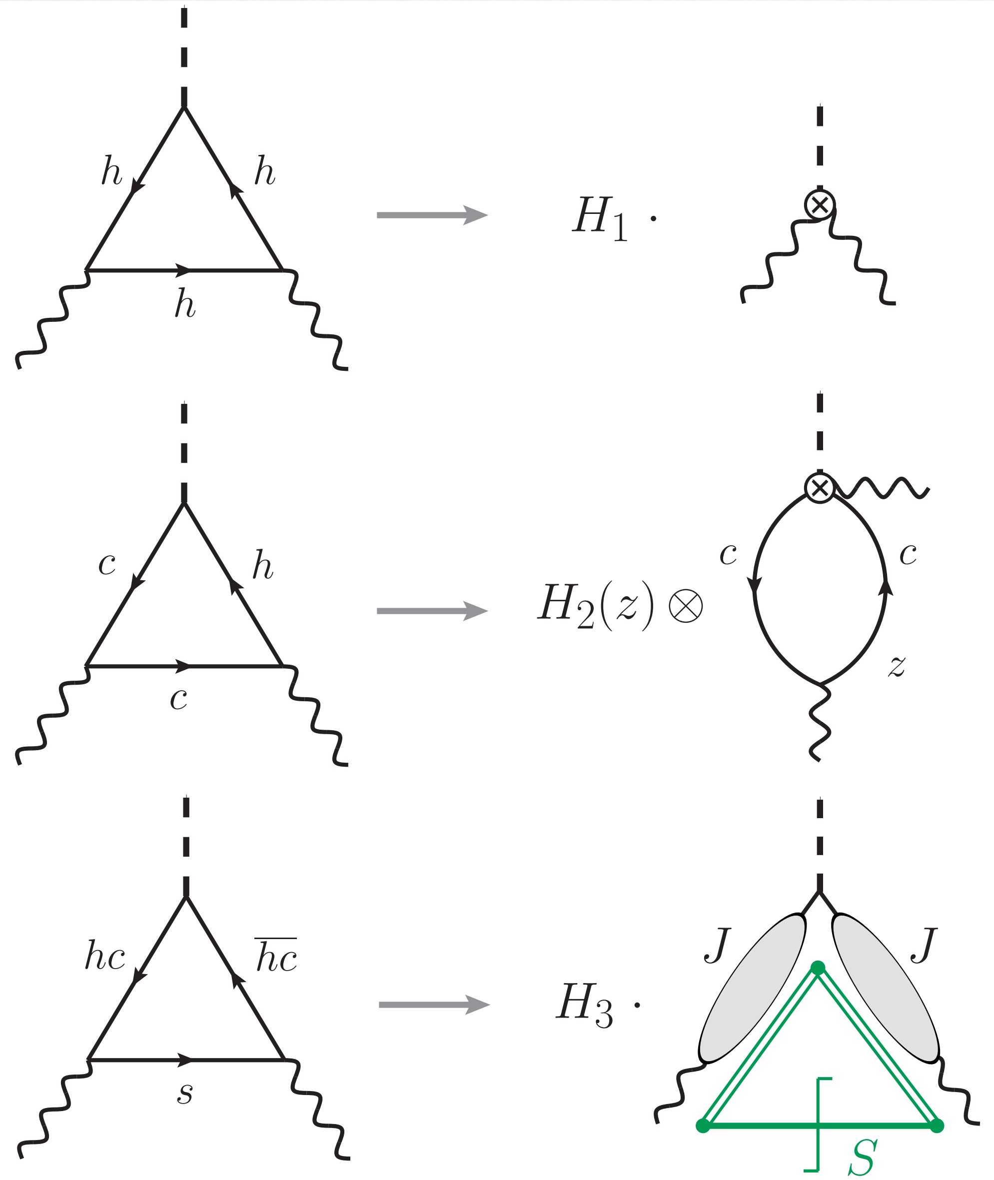
dressed collinear quark fields

$$O_3^{(0)} = T\left\{ h\, \underset{\lambda}{\bar{\mathcal{X}}_{n_1}}\, \underset{\lambda}{\mathcal{X}_{n_2}},\, i\int d^Dx\, \underset{\lambda^{1/2}}{\mathcal{L}_{q\,\xi_{n_1}}^{(1/2)}}(x),\, i\int d^Dy\, \underset{\lambda^{1/2}}{\mathcal{L}_{\xi_{n_2}q}^{(1/2)}}(y) \right\} + \text{h.c.}$$

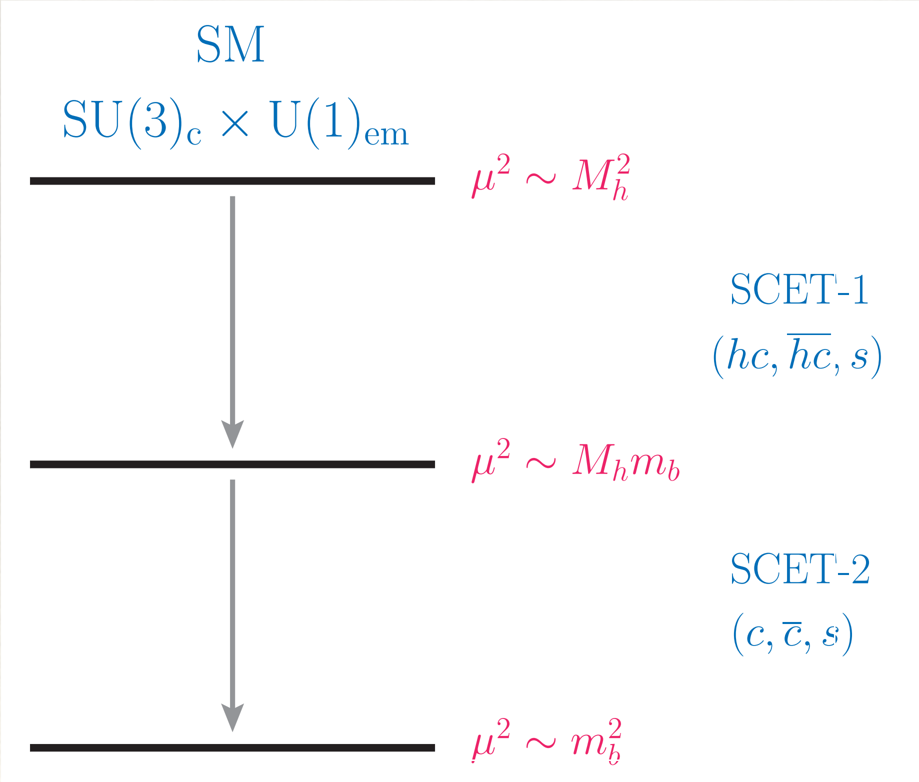
subleading SCET Lagrangian

A subleading-power observable

- Corresponding SCET operators at $O(\lambda^3)$:



Existence of only three SCET operators at $O(\lambda^3)$ ensures that these regions account for all higher-order loop graphs (see [Liu, MN 2019] for a 2-loop example)!



"Bare factorization theorem"

- Adding up the three contributions we find:

$$\mathcal{M}_b(h\to\gamma\gamma) = H_1^{(0)}\langle\gamma\gamma|\,O_1^{(0)}|h\rangle + 2\int_0^1 dz\, H_2^{(0)}(z)\,\langle\gamma\gamma|\,O_2^{(0)}(z)\,|h\rangle + H_3^{(0)}\langle\gamma\gamma|\,O_3^{(0)}|h\rangle$$

with:

$$\langle\gamma\gamma|\,O_3^{(0)}|h\rangle = \frac{g_\perp^{\mu\nu}}{2}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^\infty\frac{d\ell_-}{\ell_-}$$
$$\times\Big[J^{(0)}(M_h\ell_+)\,J^{(0)}(-M_h\ell_-) + J^{(0)}(-M_h\ell_+)\,J^{(0)}(M_h\ell_-)\Big]\,S^{(0)}(\ell_+\ell_-)$$

- Factorization formula accomplishes a naive scale separation, but all component functions are still unrenormalized!

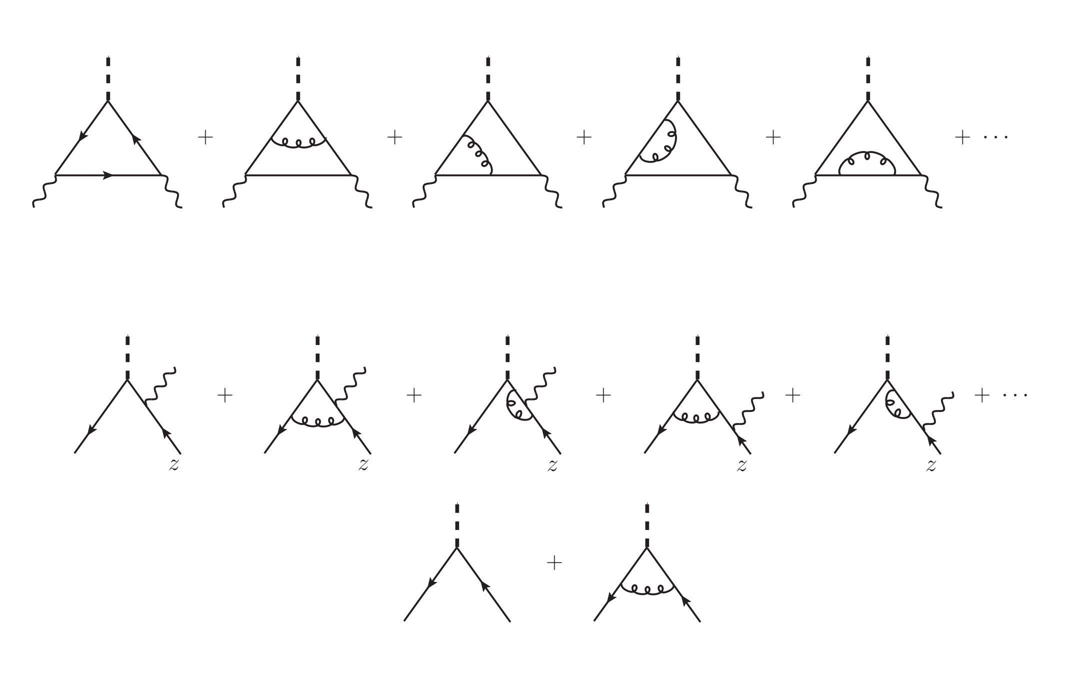
“Bare factorization theorem”

- Adding up the three contributions we find:

$$\mathcal{M}_b(h \to \gamma\gamma) = H_1^{(0)} \langle \gamma\gamma | \, O_1^{(0)} | h \rangle + 2 \int_0^1 dz \, H_2^{(0)}(z) \, \langle \gamma\gamma | \, O_2^{(0)}(z) \, | h \rangle + H_3^{(0)} \langle \gamma\gamma | \, O_3^{(0)} | h \rangle$$

- Hard matching coefficients:

$$\begin{aligned} H_1^{(0)} &= \frac{y_{b,0}}{\sqrt{2}} \frac{N_c \alpha_{b,0}}{\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} \left(1 - 3\epsilon\right) \frac{2\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(3-2\epsilon)} \\ &\quad \times \left\{ 1 - \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1+2\epsilon)\,\Gamma^2(-2\epsilon)}{\Gamma(2-3\epsilon)} \right. \\ &\qquad \times \left[\frac{2(1-\epsilon)(3 - 12\epsilon + 9\epsilon^2 - 2\epsilon^3)}{1-3\epsilon} + \frac{8}{1-2\epsilon} \frac{\Gamma(1+\epsilon)\,\Gamma^2(2-\epsilon)\,\Gamma(2-3\epsilon)}{\Gamma(1+2\epsilon)\,\Gamma^3(1-2\epsilon)} \right. \\ &\qquad\quad \left.\left. - \frac{4(3 - 18\epsilon + 28\epsilon^2 - 10\epsilon^3 - 4\epsilon^4)}{1-3\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(1+\epsilon)\,\Gamma(2-2\epsilon)} \right] \right\} \\ H_2^{(0)}(z) &= \frac{y_{b,0}}{\sqrt{2}} \left\{ \frac{1}{z} + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \right. \\ &\qquad \left. \times \left[\frac{2 - 4\epsilon - \epsilon^2}{z^{1+\epsilon}} - \frac{2(1-\epsilon)^2}{z} - 2(1 - 2\epsilon - \epsilon^2) \frac{1 - z^{-\epsilon}}{1-z} \right] \right\} + (z \to 1 - z) \\ H_3^{(0)} &= \frac{y_{b,0}}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} \, 2(1-\epsilon)^2 \frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \right] \end{aligned}$$



“Bare factorization theorem”

- Adding up the three contributions we find:

$$\mathcal{M}_b(h \to \gamma\gamma) = H_1^{(0)} \langle\gamma\gamma|\, O_1^{(0)} |h\rangle + 2 \int_0^1 dz\, H_2^{(0)}(z)\, \langle\gamma\gamma|\, O_2^{(0)}(z)\, |h\rangle + H_3^{(0)} \langle\gamma\gamma|\, O_3^{(0)} |h\rangle$$

- Operator matrix elements:

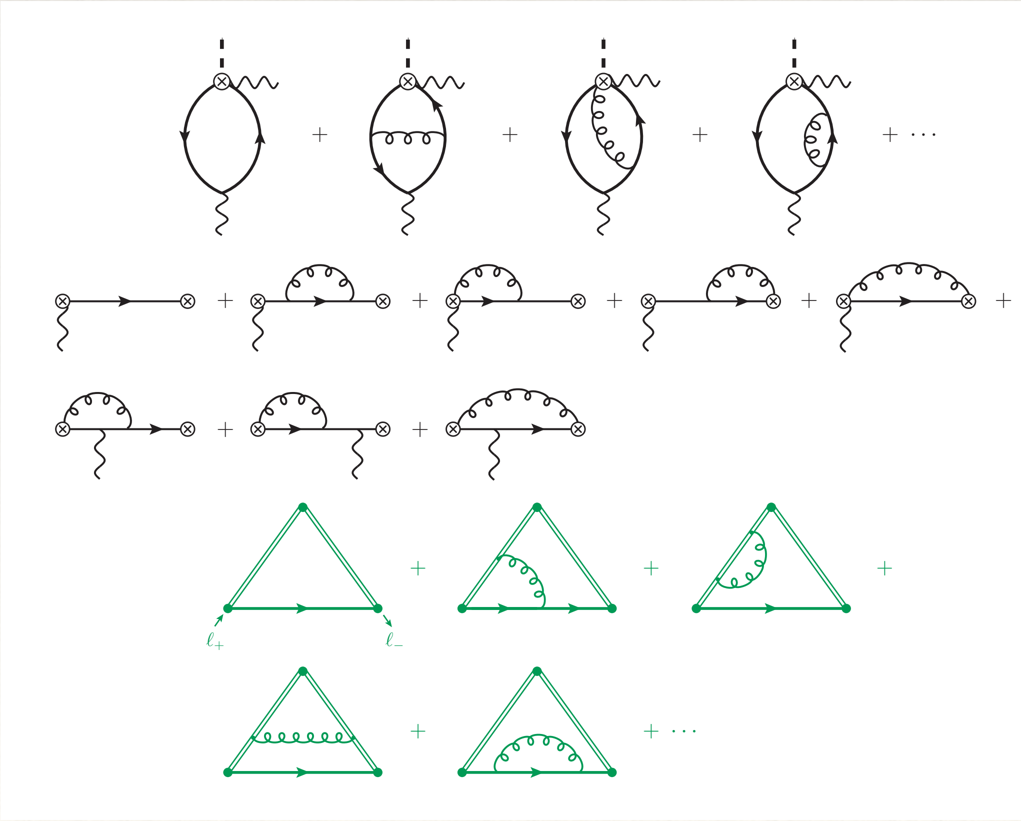
$$\langle\gamma\gamma|\, O_1^{(0)}\, |h\rangle = m_{b,0}\, g_\perp^{\mu\nu}$$

$$\langle\gamma\gamma|\, O_2^{(0)}(z)\, |h\rangle = \frac{N_c \alpha_{b,0}}{2\pi}\, m_{b,0}\, g_\perp^{\mu\nu} \left[e^{\epsilon\gamma_E}\, \Gamma(\epsilon)\, \big(m_{b,0}^2\big)^{-\epsilon} + \frac{C_F \alpha_{s,0}}{4\pi}\, \big(m_{b,0}^2\big)^{-2\epsilon} \big[K(z) + K(1-z)\big] \right]$$

$$J^{(0)}(p^2) = 1 + \frac{C_F \alpha_{s,0}}{4\pi}\, \big(-p^2 - i0\big)^{-\epsilon}\, e^{\epsilon\gamma_E}\, \frac{\Gamma(1+\epsilon)\, \Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\, (2 - 4\epsilon - \epsilon^2)$$

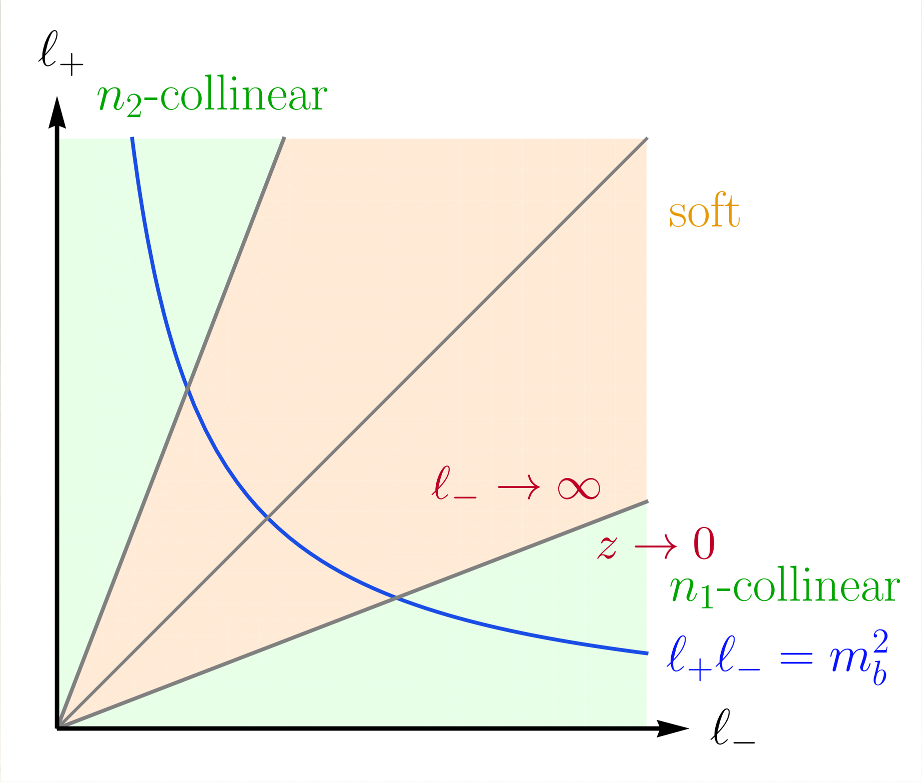
$$S^{(0)}(w) = -\frac{N_c \alpha_{b,0}}{\pi}\, m_{b,0} \left[S_a^{(0)}(w)\, \theta(w - m_{b,0}^2) + S_b^{(0)}(w)\, \theta(m_{b,0}^2 - w) \right]$$

$$\dots$$



Endpoint divergences

- Closer inspection shows that the convolution integrals in the factorization formula are divergent for $z \to 0, 1$ (second term) and $\ell_\pm \to \infty$ (third term)
- Second term is symmetric under $z \leftrightarrow (1-z)$ and it suffices to study the singularity at $z \to 0$
- Physical origin: **overlap of soft and collinear regions**, whose boundaries are not separated by the dimensional regulator



Endpoint divergences

- In order to define the two convolutions properly one needs to introduce a **rapidity regulator** under the integrals:

$$\mathcal{M}_b(h \to \gamma\gamma) = \lim_{\eta\to 0} H_1^{(0)} \langle \gamma\gamma | O_1^{(0)} | h \rangle + 4 \int_0^1 \frac{dz}{z} \left(\frac{-zM_h^2 - i0}{\nu^2} \right)^\eta \bar{H}_2^{(0)}(z) \, \langle \gamma\gamma | \, O_2^{(0)}(z) \, | h \rangle$$

$$+ g_\perp^{\mu\nu} H_3^{(0)} \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^{\ell_-} \frac{d\ell_+}{\ell_+} S^{(0)}(\ell_+ \ell_-)$$

$$\times \left[\left(\frac{\bar{n}_2 \cdot k_2 \, \ell_- - i0}{\nu^2} \right)^\eta J^{(0)}(\bar{n}_1 \cdot k_1 \, \ell_+) \, J^{(0)}(-\bar{n}_2 \cdot k_2 \, \ell_-) \right.$$

$$\left. + \left(\frac{-\bar{n}_2 \cdot k_2 \, \ell_- - i0}{\nu^2} \right)^\eta J^{(0)}(-\bar{n}_1 \cdot k_1 \, \ell_+) \, J^{(0)}(\bar{n}_2 \cdot k_2 \, \ell_-) \right]$$

- Endpoint divergences lead to $1/\eta$ poles, which cancel in the sum of all terms!

Endpoint divergences

- Things are, in fact, even more subtle. For example, in higher orders one finds that:

$$\bar{H}_2^{(0)}(z) \sim z^{-n\epsilon} \qquad \text{but} \qquad \langle O_2^{(0)}(z)\rangle \sim z^{+m\epsilon}$$

- Terms with $m = n$ require the rapidity regulator when integrated over $\int_0^1 \frac{dz}{z}$, while those with $m \neq n$ are regularized by the dimensional regulator

- In simpler examples based on SCET-1, the dimensional regulator regularizes all endpoint divergences, but this still leaves the problem of how to deal with the $1/\epsilon$ poles from the endpoint singularities, which spoil factorization

[Beneke et al., Moult et al. 2018-2020]

Endpoint divergences

- All-order cancellation of $1/\eta$ poles requires that the integrands of the second and third term are the same when evaluated in the singular regions!

- This is ensured by the D-dim. **refactorization conditions**:

[Liu, MN 2019]

$$[\![\bar{H}_2^{(0)}(z)]\!] = -H_3^{(0)} J^{(0)}(zM_h^2)$$

$$[\![\langle\gamma\gamma|\, O_2^{(0)}(z)\, |h\rangle]\!] = -\frac{g_\perp^{\mu\nu}}{2} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J^{(0)}(-M_h\ell_+)\, S^{(0)}(zM_h\ell_+)$$

- We have recently proved these relations using SCET tools:

[2009.06779]

Removing endpoint divergences

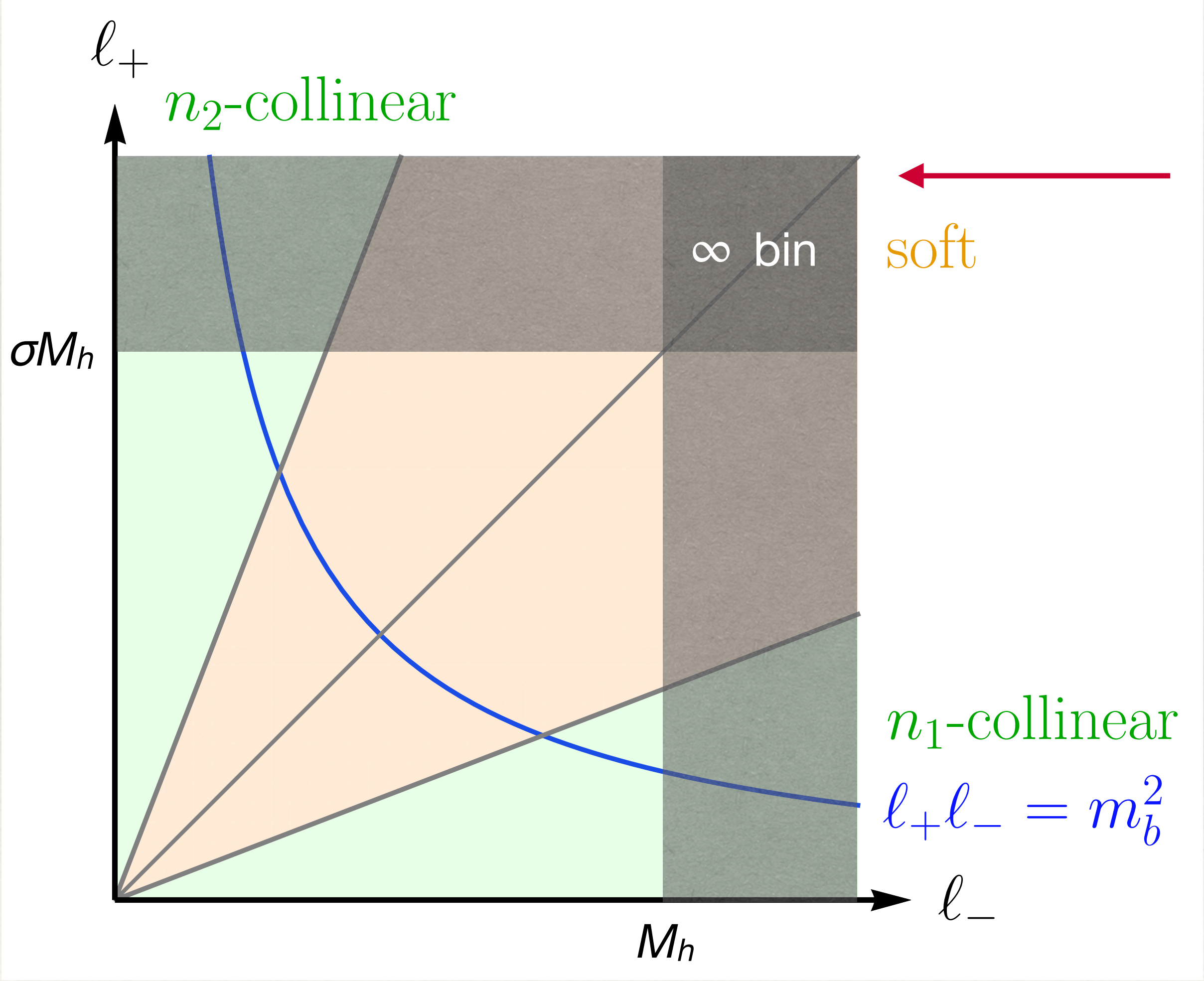
- Using these relations, the bare factorization formula can be rearranged in such a way that all endpoint divergences are removed and the limit $\eta\to 0$ can be taken. We find:

integrand for $z\to 0$

$$
\begin{aligned}
\mathcal{M}_b &= \left(H_1^{(0)} + \Delta H_1^{(0)}\right) \langle\gamma\gamma|\,O_1^{(0)}|h\rangle \\
&\quad + 2\lim_{\delta\to 0}\int_\delta^{1-\delta} dz \left[H_2^{(0)}(z)\,\langle\gamma\gamma|\,O_2^{(0)}(z)\,|h\rangle - \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z}\,[\![\langle\gamma\gamma|\,O_2^{(0)}(z)\,|h\rangle]\!] \right.\\
&\qquad \left. - \frac{[\![\bar{H}_2^{(0)}(1-z)]\!]}{1-z}\,[\![\langle\gamma\gamma|\,O_2^{(0)}(1-z)\,|h\rangle]\!]\right] \\
&\quad + g_\perp^{\mu\nu}\lim_{\sigma\to -1} H_3^{(0)} \int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J^{(0)}(M_h\ell_-)\,J^{(0)}(-M_h\ell_+)\,S^{(0)}(\ell_+\ell_-)\Big|_{\text{leading power}}
\end{aligned}
$$

Removing endpoint divergences

- In the space of momentum modes, this amount to the following subtractions in the third term:



"infinity bin" is subtracted twice and must be added back as a hard contribution $\Delta H_1^{(0)}$ to the coefficient of the first term

Renormalized factorization theorem

- So far, the factorization formula is still expressed in terms of bare quantities, but we wish to establish a corresponding renormalized formula:

$$\begin{aligned}\mathcal{M}_b &= H_1(\mu)\,\langle O_1(\mu)\rangle \\ &\quad + 2\int_0^1 dz\,\bigg[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\langle O_2(\bar{z},\mu)\rangle]\!]\bigg] \\ &\quad + g_\perp^{\mu\nu}\,H_3(\mu)\,\lim_{\sigma\to-1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu)\Big|_{\text{leading power}}\end{aligned}$$

- This is non-trivial, because the presence of cutoffs does not commute with renormalization!

Renormalized factorization theorem

- Renormalization conditions for the operators:

$$O_1(\mu) = Z_{11}\, O_1^{(0)}$$

$$O_2(z,\mu) = \int_0^1 dz'\, Z_{22}(z,z')\, O_2^{(0)}(z') + Z_{21}(z)\, O_1^{(0)}$$

[2009.06779]

incompatible with cutoffs

$$[\![O_2(z,\mu)]\!] = \int_0^{\infty} dz'\, [\![Z_{22}(z,z')]\!]\, [\![O_2^{(0)}(z')]\!] + [\![Z_{21}(z)]\!]\, O_1^{(0)}$$

incompatible with cutoffs

$$J(p^2,\mu) = \frac{1}{p^2} \int_0^{\infty} dp'^{\,2}\, Z_J(p^2,p'^{\,2};\mu)\, J^{(0)}(p'^{\,2})$$

[2003.03393]

incompatible with cutoffs

$$S(w,\mu) = \int_0^{\infty} dw'\, Z_S(w,w';\mu)\, S^{(0)}(w')$$

[2005.03013]

with complicated Z factors containing plus distributions

Renormalized factorization theorem

- When the cutoffs are move from the bare over to the renormalized functions, some **left-over terms** remain, which individually have a rather complicated structure and depend both on the hard scale M_h and the soft scale m_b
- The most non-trivial part of the derivation of the renormalized factorization theorem was to show that, to all orders of perturbation theory, the sum of the left-over terms takes the form of an **additional hard subtraction** $\delta H_1^{(0)}$ of the Wilson coefficient of the operator $O_1^{(0)}$ [2009.06779]

Renormalized factorization theorem

- After this crucial step had been accomplished, we could derive the renormalization conditions for the matching coefficients:

$$H_1(\mu) = \Big(H_1^{(0)} + \Delta H_1^{(0)} - \delta H_1^{(0),\,\text{tot}} \Big)\, Z_{11}^{-1}$$

$$+\, 2 \lim_{\delta\to 0} \int_\delta^{1-\delta} dz \left[H_2^{(0)}(z)\, Z_{21}^{-1}(z) - \frac{[\![\bar H_2^{(0)}(z)]\!]}{z} [\![Z_{21}^{-1}(z)]\!] - \frac{[\![\bar H_2^{(0)}(\bar z)]\!]}{\bar z} [\![Z_{21}^{-1}(\bar z)]\!] \right]$$

$$H_2(z,\mu) = \int_0^1 dz'\, H_2^{(0)}(z')\, Z_{22}^{-1}(z',z)$$

$$\frac{[\![\bar H_2(z,\mu)]\!]}{z} = \int_0^\infty dz'\, \frac{[\![\bar H_2^{(0)}(z')]\!]}{z'} [\![Z_{22}^{-1}(z',z)]\!]$$

$$H_3(\mu) = H_3^{(0)} Z_{33}^{-1}$$

Renormalized factorization theorem

- Renormalized matrix elements, with $L_m = \ln(m_b^2/\mu^2)$:

$$\langle O_1(\mu)\rangle = m_b(\mu)\, g_\perp^{\mu\nu}$$

$$\langle O_2(z,\mu)\rangle = \frac{N_c\alpha_b}{2\pi}\, m_b(\mu)\, g_\perp^{\mu\nu} \Big\{ - L_m + \frac{C_F\alpha_s}{4\pi} \Big[L_m^2 \Big(\ln z + \ln(1-z) + 3 \Big) - L_m \Big(\ln^2 z + \ln^2(1-z) - 4\ln z \ln(1-z) + 11 - \frac{2\pi^2}{3} \Big) + F(z) + F(1-z) \Big] + \mathcal{O}(\alpha_s^2) \Big\}$$

$$J(p^2,\mu) = 1 + \frac{C_F\alpha_s}{4\pi} \Big[\ln^2\Big(\frac{-p^2 - i0}{\mu^2} \Big) - 1 - \frac{\pi^2}{6} \Big] + \mathcal{O}(\alpha_s^2)$$

$$S(w,\mu) = -\frac{N_c\alpha_b}{\pi}\, m_b(\mu) \Big[S_a(w,\mu)\, \theta(w - m_b^2) + S_b(w,\mu)\, \theta(m_b^2 - w) \Big]$$

$$S_a(w,\mu) = 1 + \frac{C_F\alpha_s}{4\pi} \Big[- L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + 2\,\mathrm{Li}_2\Big(\frac{1}{\hat{w}}\Big) - 4\ln\Big(1 - \frac{1}{\hat{w}}\Big) \Big(L_m + 1 + \ln\Big(1 - \frac{1}{\hat{w}}\Big) + \frac{3}{2}\ln\hat{w} \Big) \Big] + \mathcal{O}(\alpha_s^2)$$

$L_w = \ln(w/\mu^2)$
$\hat{w} = w/m_b^2$

Renormalized factorization theorem

- Renormalized matrix elements, with $L_m = \ln(m_b^2/\mu^2)$:

$$\langle O_1(\mu)\rangle = m_b(\mu)\, g_\perp^{\mu\nu}$$

$$\langle O_2(z,\mu)\rangle = \frac{N_c\alpha_b}{2\pi}\, m_b(\mu)\, g_\perp^{\mu\nu} \bigg\{ - L_m + \frac{C_F\alpha_s}{4\pi} \bigg[L_m^2 \Big(\ln z + \ln(1-z) + 3 \Big) - L_m \left(\ln^2 z + \ln^2(1-z) - 4\ln z \ln(1-z) + 11 - \frac{2\pi^2}{3} \right) + F(z) + F(1-z) \bigg] + \mathcal{O}(\alpha_s^2) \bigg\}$$

$$J(p^2,\mu) = 1 + \frac{C_F\alpha_s}{4\pi} \left[\ln^2\!\left(\frac{-p^2 - i0}{\mu^2} \right) - 1 - \frac{\pi^2}{6} \right] + \mathcal{O}(\alpha_s^2)$$

$$S(w,\mu) = -\frac{N_c\alpha_b}{\pi}\, m_b(\mu) \Big[S_a(w,\mu)\, \theta(w - m_b^2) + S_b(w,\mu)\, \theta(m_b^2 - w) \Big]$$

$$L_w = \ln(w/\mu^2)$$
$$\hat{w} = w/m_b^2$$

$$S_b(w,\mu) = \frac{C_F\alpha_s}{\pi} \ln(1-\hat{w}) \Big[L_m + \ln(1-\hat{w}) \Big] + \mathcal{O}(\alpha_s^2)$$

Renormalized factorization theorem

- Renormalized matching coefficients, with $L_h = \ln(-M_h^2/\mu^2)$:

$$H_1(\mu) = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \left\{ -2 + \frac{C_F \alpha_s}{4\pi} \left[-\frac{\pi^2}{3} L_h^2 + (12 + 8\zeta_3) L_h - 36 - \frac{2\pi^2}{3} - \frac{11\pi^4}{45} \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$H_2(z,\mu) = \frac{y_b(\mu)}{\sqrt{2}} \frac{1}{z(1-z)} \left\{ 1 + \frac{C_F \alpha_s}{4\pi} \left[2L_h \big(\ln z + \ln(1-z) \big) + \ln^2 z + \ln^2(1-z) - 3 \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$H_3(\mu) = \frac{y_b(\mu)}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_s}{4\pi} \left(L_h^2 + 2 - \frac{\pi^2}{6} \right) + \mathcal{O}(\alpha_s^2) \right]$$

Resummation of large logarithms

Liu, Mecaj, MN, Yang: 2009.04456 & 2009.06779
Liu, MN: 2003.03393 (JHEP)
Liu, Mecaj, MN, Yang, Fleming: 2005.03013 (JHEP)

Resummation of large logs

- The renormalized factorization formula

$$
\begin{aligned}
\mathcal{M}_b &= H_1(\mu)\,\langle O_1(\mu)\rangle \\
&\quad + 2\int_0^1 dz \left[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\langle O_2(\bar{z},\mu)\rangle]\!] \right] \\
&\quad + g_\perp^{\mu\nu}\,H_3(\mu)\,\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu)\Big|_{\text{leading power}}
\end{aligned}
$$

provides a complete scale separation and allows us to resum large logarithms in the decay amplitude to all orders of perturbation theory!

Resummation of large logs

- RG equations for matrix elements:

$$\frac{d}{d\ln\mu}\langle O_1(\mu)\rangle = -\gamma_{11}\langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu}\langle O_2(z,\mu)\rangle = -\int_0^1 dz'\,\gamma_{22}(z,z')\,\langle O_2(z',\mu)\rangle - \gamma_{21}(z)\,\langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu}J(p^2,\mu) = -\int_0^\infty dx\,\gamma_J(p^2,xp^2)\,J(xp^2,\mu)$$

$$\frac{d}{d\ln\mu}S(w,\mu) = -\int_0^\infty dx\,\gamma_S(w,w/x)\,S(w/x,\mu)$$

Resummation of large logs

- RG equations for matching coefficients:

inhomogeneous contribution due to cutoffs

$$\frac{d}{d\ln\mu} H_1(\mu) = D_{\rm cut}(\mu) + \gamma_{11}\, H_1(\mu)$$

$$+ 2\int_0^1 dz \left[H_2(z,\mu)\,\gamma_{21}(z) - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\, [\![\gamma_{21}(z)]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\, [\![\gamma_{21}(\bar{z})]\!] \right]$$

$$\frac{d}{d\ln\mu} H_2(z,\mu) = \int_0^1 dz'\, H_2(z',\mu)\,\gamma_{22}(z',z)$$

$$\frac{d}{d\ln\mu} H_3(\mu) = \gamma_{33}\, H_3(\mu)$$

- where:

$$D_{\rm cut}(\mu) = -\frac{N_c \alpha_b}{\pi}\, \frac{y_b(\mu)}{\sqrt{2}} \left[\frac{C_F \alpha_s}{4\pi}\, 16\zeta_3 + \left(\frac{\alpha_s}{4\pi} \right)^2 d_{{\rm cut},2} + \mathcal{O}(\alpha_s^3) \right] \ni \alpha_b\, (\alpha_s L_h)^n$$

Logarithms in the 3-loop amplitude

- From a perturbative solution of the RGEs, we have obtained predictions for the terms of order $\mathcal{O}(\alpha_s^2 L^k)$ with k=6,5,4,3 in the 3-loop decay amplitude in the on-shell scheme, finding:

$$\begin{aligned}
\mathcal{M}_b &= \frac{N_c \alpha_b}{\pi} \frac{m_b^2}{v} \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2) \\
&\times \Bigg\{ \frac{L^2}{2} - 2 + \frac{C_F \alpha_s(\hat\mu_h)}{4\pi} \left[-\frac{L^4}{12} - L^3 - \frac{2\pi^2}{3} L^2 + \left(12 + \frac{2\pi^2}{3} + 16\zeta_3 \right) L - 20 + 4\zeta_3 - \frac{\pi^4}{5} \right] \\
&\quad + C_F \left(\frac{\alpha_s(\hat\mu_h)}{4\pi} \right)^2 \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} - \frac{\beta_0}{30} \right) L^5 + d_4^{\rm OS} L^4 + d_3^{\rm OS} L^3 + \dots \right] \Bigg\}
\end{aligned}$$

- Find perfect agreement with recent numerical results!

[Czakon, Niggetiedt 2020]

Series of subleading logs

- We have reproduced the series of leading double logs (LL) and obtained a **new result** for the NLL logs to all orders in α_s:

$$\mathcal{M}_b^{\rm NLL} = \frac{N_c \alpha_b}{\pi} \frac{y_b(M_h)}{\sqrt{2}} m_b \, \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2) \, \frac{L^2}{2} \sum_{n=0}^\infty (-\rho)^n \frac{2\Gamma(n+1)}{\Gamma(2n+3)}$$

[Kotsky, Yakovlev 1997]

$$\times \left[1 + \frac{3\rho}{2L} \frac{2n+1}{2n+3} - \frac{\beta_0}{C_F} \frac{\rho^2}{4L} \frac{(n+1)^2}{(2n+3)(2n+5)} \right],$$

with $\rho = \dfrac{C_F \alpha_s(M_h)}{2\pi} L^2$

- The subleading terms disagree with earlier results in the literature!

[Akhoury, Wang, Yakovlev 2001; Anastasiou, Penin 2020]

Resummation in RG-improved PT

- Ultimate goal is to resum **all large logarithms** and exponentiate them (RG-improved perturbation theory)
- Particularly important for Sudakov problems, where leading logs are formally larger than $O(1)$
- In RG-improved perturbation theory one supplies the matching conditions for all component functions in the factorization theorem at matching scales where they are free of large logs; these functions and then evolved to a common scale solving their RG equations → **all large logs exponentiate**!

Resummation in RG-improved PT

- We have not yet performed a complete resummation, but we have resummed the most difficult contribution T_3 at LO in RG-improved perturbation theory, finding: [2009.04456]

$$T_3^{\rm LO} = \frac{\alpha}{3\pi}\,\frac{y_b(\mu_h)}{\sqrt{2}} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{M_h} \frac{d\ell_+}{\ell_+}\, m_b(\mu_s)\, e^{2\mathcal{S}(\mu_s,\mu_h)-2\mathcal{S}(\mu_-,\mu_h)-2\mathcal{S}(\mu_+,\mu_h)} \left(\frac{-M_h\ell_-}{\mu_-^2}\right)^{a_\Gamma^-} \left(\frac{-M_h\ell_+}{\mu_+^2}\right)^{a_\Gamma^+} \left(\frac{-\ell_+\ell_-}{\mu_s^2}\right)^{-a_\Gamma^s}$$
$$\times \left(\frac{\alpha_s(\mu_s)}{\alpha_s(\mu_h)}\right)^{\frac{12}{23}} e^{-2\gamma_E\, a_\Gamma^+} \frac{\Gamma(1-a_\Gamma^+)}{\Gamma(1+a_\Gamma^+)}\, e^{-2\gamma_E\, a_\Gamma^-} \frac{\Gamma(1-a_\Gamma^-)}{\Gamma(1+a_\Gamma^-)}\, e^{4\gamma_E\, a_\Gamma^s}\, G_{4,4}^{2,2}\left(\begin{matrix} -a_\Gamma^s\,, & -a_\Gamma^s\,, & 1-a_\Gamma^s\,, & 1-a_\Gamma^s \\ 0\,, & 1\,, & 0\,, & 0 \end{matrix} \,\middle|\, \frac{m_b^2}{-\ell_+\ell_-} \right)$$

with:

$$a_\Gamma^i = -\frac{8}{23}\ln\frac{\alpha_s(\mu_i)}{\alpha_s(\mu_h)}\,, \qquad \mathcal{S}(\mu_i,\mu_h) = \frac{12}{529}\left[\frac{4\pi}{\alpha_s(\mu_i)}\left(1-\frac{1}{r}-\ln r\right) + \frac{58}{23}\ln^2 r + \left(\frac{2429}{207}-\pi^2\right)(1-r+\ln r)\right]$$

$r = \alpha_s(\mu_h)/\alpha_s(\mu_i)$

- dynamical matching scales:

$$\mu_s^2 \sim \ell_+\ell_- \qquad \mu_\pm^2 \sim M_h\ell_\pm \qquad \mu_h \sim M_h$$

Conclusions

- We have derived the first SCET factorization theorem for an observable appearing at subleading order in power counting
- Generic features:
  - several SCET operators exist → several terms in factorization formula
  - these operators mix under renormalization
  - endpoint divergences in convolutions cancel between the different terms; cancellation ensured by ***D*-dim. refactorization conditions**
  - endpoint divergences can be removed by performing subtractions and rearranging the various terms
- Our results are an important step towards establishing SCET as a complete EFT admitting a consistent power expansion!