Modular Symmetry in Flavors

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Outline of my talk

1 Introduction

2 Towards Non-Abelian Flavor Symmetry

3 Prototype of Flavor model with A_4

4 Modular Group

5 Predictions in Modular A_4 Symmetry

6 Modular S_3 and S_4 Symmetries

7 Summary

1 Introduction

We have a big question since the discovery of Muon

"Who orderd that ?" 1937 Isidor Issac Rabi

What is the principle to control flavors of quarks/leptons ?

The precise measurements of CKM mixing angles and CP violating phase of quarks established the SM (3 families).

Now, the neutrino oscillation experiments are going on observation of lepton mixing angles precisely.
Furthremore, CP violation of lepton sector is within reach
@T2K and Nova experiments T2HK, DUNE.

It may be an important clue for Beyond SM (flavor).

In the beginning of 21th century, neutrino oscillation experiments presented the lepton mixing $\sin^2\theta_{12}\sim 1/3$, $\sin^2\theta_{23}\sim 1/2$.

no data for θ_{13}

Harrison, Perkins, Scott (2002) proposed

Tri-bimaximal Mixing of Neutrino flavors.

$$\sin^2\theta_{12} = 1/3,\ \sin^2\theta_{23} = 1/2,\ \sin^2\theta_{13} = 0,$$

$$\boxed{U_{\text{tri-bimaximal}} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}}$$

Tri-bimaximal Mixing (TBM) is realized by the mass matrix

$$m_{TBM} = \frac{m_1+m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

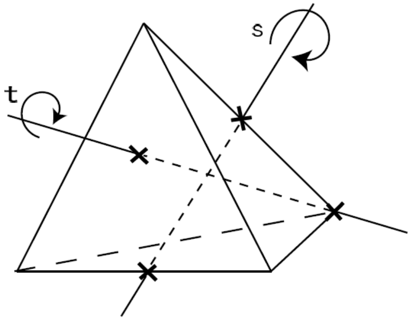
in the diagonal basis of charged leptons.

A_4 symmetric

Integer (inter-family related) matrix elements suggest Non-Abelian Discrete Flavor Symmetry.

E. Ma, G. Rajasekaran 2001

A_4 group



Symmetry of tetrahedron

Even permutation group of four objects (1234)
12 elements (order 12) are generated by
S and T: $S^2=T^3=(ST)^3=1$: S=(14)(23), T=(123)

4 conjugacy classes

C1: 1 h=1
C3: S, T^2ST, TST^2 h=2
C4: T, ST, TS, STS h=3
C4': T^2, ST^2, T^2S, ST^2S h=3

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Irreducible representations: 1, 1', 1", 3
The minimum group containing triplet without doublet.

Multiplication rule of A_4 group

Irreducible representations: 1, 1', 1", 3

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

for triplet

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = \boxed{(a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}}} \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{\mathbf{1'}}$$

$$\oplus (a_2 b_2 + a_1 b_3 + a_3 b_1)_{\mathbf{1''}}$$

$$\oplus \frac{1}{3}\begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}}$$

A_4 invariant Majorana neutrino mass term

$$(LL)_1 = L_1 L_1 + L_2 L_3 + L_3 L_2$$

3 x 3

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$$

A_4 invariant

In 2012, θ_{13} was measured by Daya Bay, RENO, Double Chooz, T2K, MINOS,

Tri-bimaximal mixing was ruled out !

$$\theta_{13} \simeq 9^\circ \simeq \theta_c/\sqrt{2}$$

Rather large θ_{13} promoted to search for *CP* violation !

$$J_{CP} = s_{23}c_{23}s_{12}c_{12}s_{13}c_{13}^2 \sin\delta_{CP} \simeq 0.0327 \sin\delta$$

J_{CP}(quark)$\sim 3\times10^{-5}$

CP violating phase δ_{CP} is a key parameter to understand flavors as well as two large mixing angles θ_{12} and θ_{23}.

Neutrino mixing matrix

$$\nu_{\alpha} = (U_{\mathrm{PMNS}})_{\alpha i}\, \nu_{i}$$

α =e, μ , τ

i=1, 2, 3

flavor eigenstates

mass eigenstates

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

c_{ij} and s_{ij} denote $\cos\theta_{ij}$ and $\sin\theta_{ij}$, respectively.

$m_1 < m_2 < m_3$

$m_3 < m_1 < m_2$

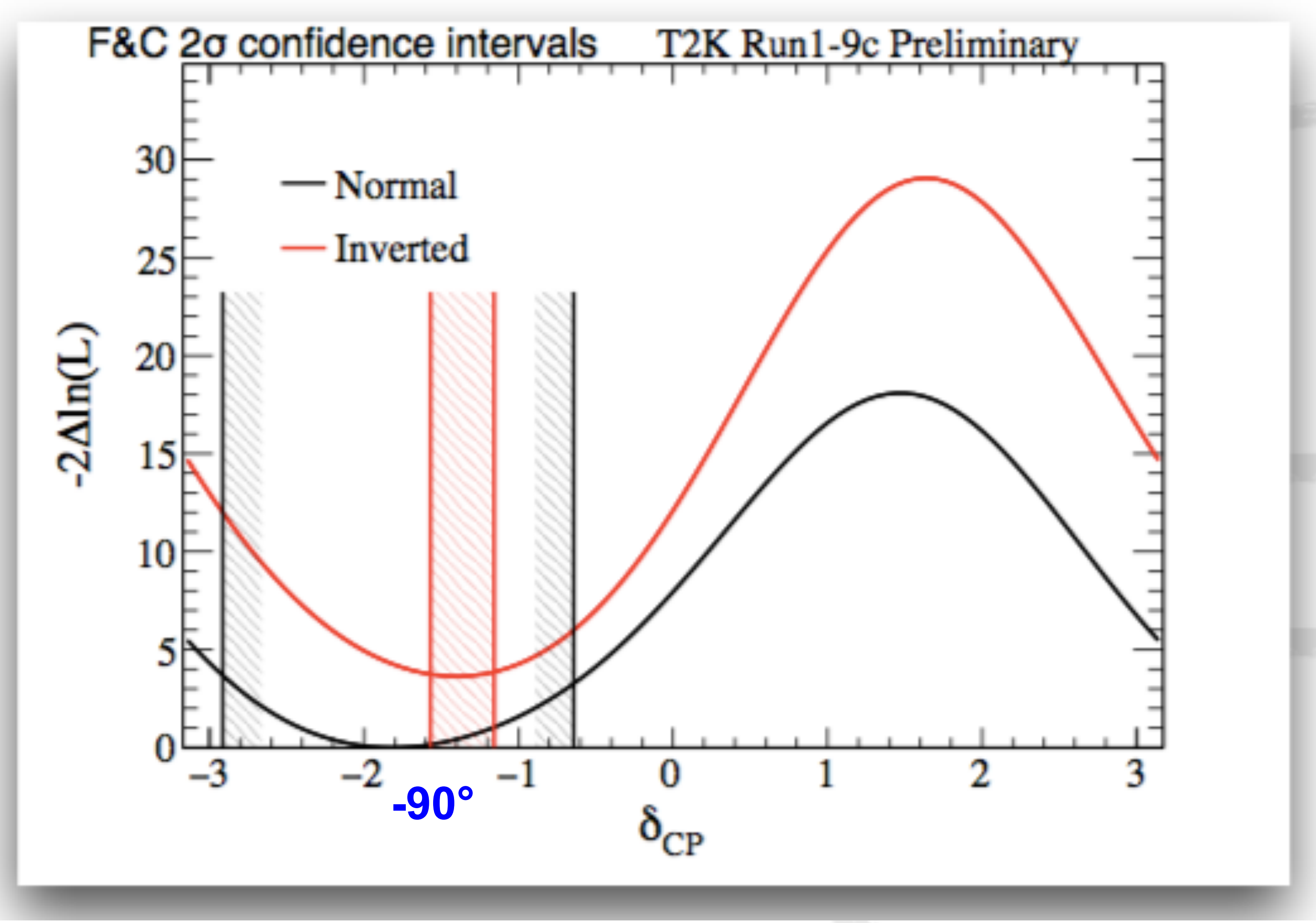
observable	$3\,\sigma$ range for NH	$3\,\sigma$ range for IH
$\Delta m^2_{\mathrm{atm}}$	$(2.399 \sim 2.593) \times 10^{-3}\mathrm{eV}^2$	$(-2.562 \sim -2.369) \times 10^{-3}\mathrm{eV}^2$
$\Delta m^2_{\mathrm{sol}}$	$(6.80 \sim 8.02) \times 10^{-5}\mathrm{eV}^2$	$(6.80 \sim 8.02) \times 10^{-5}\mathrm{eV}^2$
$\sin^2\theta_{23}$	$0.418 \sim 0.613$	$0.435 \sim 0.616$
$\sin^2\theta_{12}$	$0.272 \sim 0.346$	$0.272 \sim 0.346$
$\sin^2\theta_{13}$	$0.01981 \sim 0.02436$	$0.02006 \sim 0.02452$

NuFIT3.2 (2018)

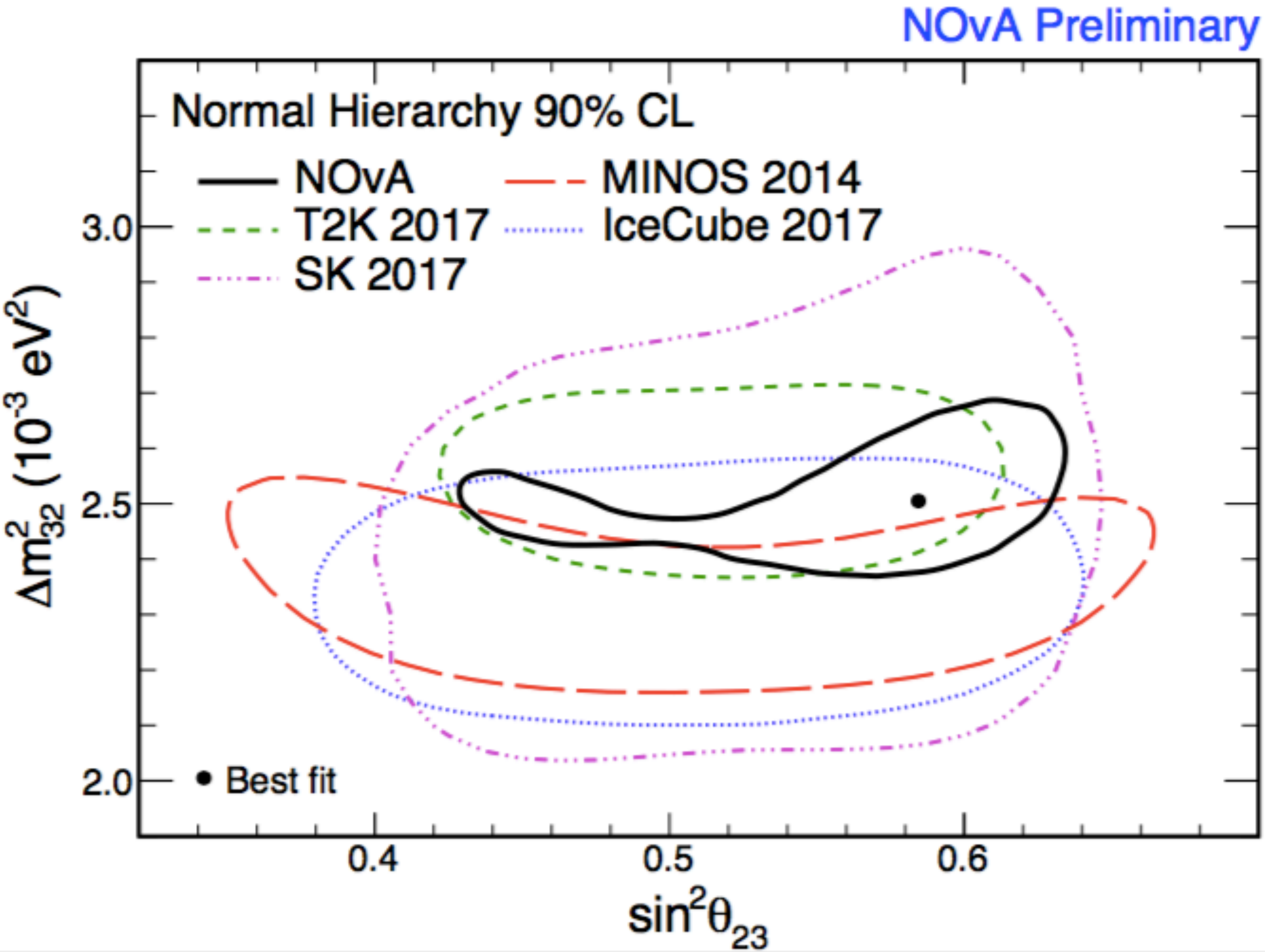
$\Delta m^2_{\mathrm{atm}} = m_3^2 - m_1^2$, $\Delta m^2_{\mathrm{sol}} = m_2^2 - m_1^2$

Neutrino 2018

DATA FIT with reactor constraint



- **CP conserving values of δ_{CP} lie outside 2σ region.**


NOvA Preliminary
Normal Hierarchy 90% CL
NOvA
MINOS 2014
T2K 2017
IceCube 2017
SK 2017
Best fit
Δm^2_{32} (10^{-3} eV^2)
3.0
2.5
2.0
0.4
0.5
0.6
$\sin^2\theta_{23}$

If θ_{23} is rather less than 45° it could be related neutrino masses.

For example,

$$\boxed{\sin^2\theta_{23} \simeq \sqrt[4]{\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}} = 0.40 \sim 0.43}$$

FTY(2003), FSTY(2012)

Just like GST relation

GST 1968 Weinberg 1977

$$M_{\rm d} = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \quad \Rightarrow \quad \theta_{12} \simeq \sqrt{\frac{m_d}{m_s}}$$

However, the closer θ_{23} =45° or > 45° the more likely that some symmetry/structure behind it.

2 Towards Non-Abelian Flavor Symmetry

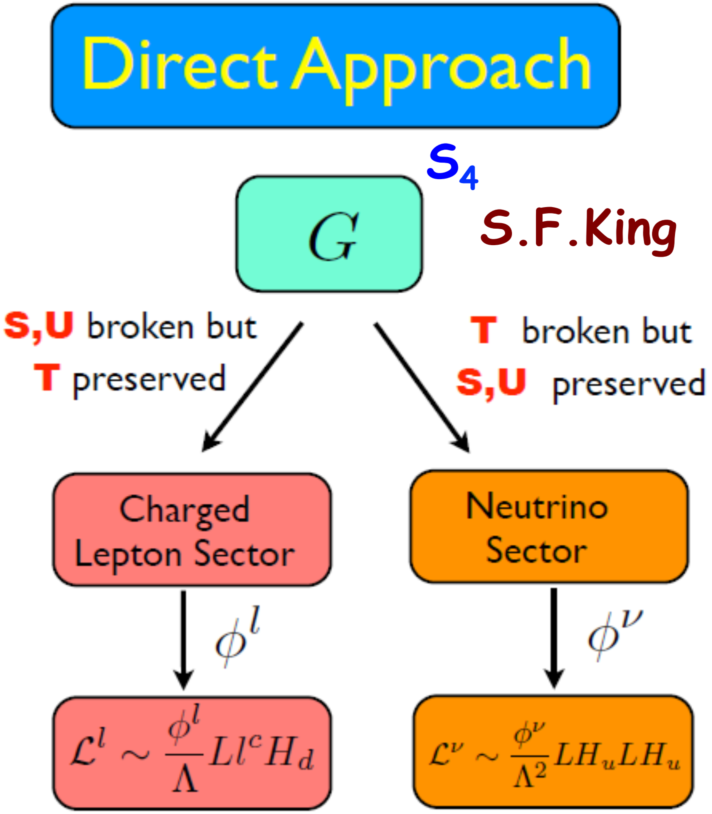
Footprint of the non-Abelian discrete symmetry is expected to be seen in the neutrino mixing matrix.

How to find an imprint of generators of finite groups

Generators of G (S,T,U) determine the flavor mixing directly.

Suppose group G for flavors at high energy.

At low energy, different subgroups of G are preserved in Yukawa sectors of Neutrinos and Charged leptons, respectively.



Direct Approach

S_4

G

S.F.King

S,U broken but **T** preserved

T broken but **S,U** preserved

Charged Lepton Sector

Neutrino Sector

ϕ^l

ϕ^ν

$\mathcal{L}^l \sim \frac{\phi^l}{\Lambda} L l^c H_d$

$\mathcal{L}^\nu \sim \frac{\phi^\nu}{\Lambda^2} L H_u L H_u$

Consider the case of A_4 flavor symmetry:

A_4 has subgroups:

three Z_2, four Z_3, one $Z_2 \times Z_2$ (klein four-group)

$S^2=T^3=(ST)^3=1$

Z_2: {1,S}, {1,T^2ST}, {1,TST^2}
Z_3: {1,T,T^2}, {1,ST,T^2S}, {1,TS, ST^2}, {1,STS,ST^2S}
K_4: {1,S,T^2ST,TST^2}

Suppose A_4 is spontaneously broken to one of subgroups:
Neutrino sector preserves Z_2: {1,S}
Charged lepton sector preserves Z_3: {1,T,T^2}

$$S^T m^{\nu}_{LL} S = m^{\nu}_{LL}, \quad T^{\dagger} Y_e Y_e^{\dagger} T = Y_e Y_e^{\dagger}$$

$$\Downarrow$$

$$[S, m^{\nu}_{LL}] = 0, \quad [T, Y_e Y_e^{\dagger}] = 0$$

Mixing matrices diagonalise m^{ν}_{LL}, $Y_e Y_e^{\dagger}$ also diagonalize S and T, respectively !

For the triplet, the representations are given as

$$S=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix},\quad T=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega\end{pmatrix};\quad \omega=e^{2\pi i/3}$$

$$V_\nu^T S V_\nu=\mathrm{diag}(\enclose{circle}{-1},1,\enclose{circle}{-1})$$

$$V_\nu=\begin{pmatrix}2/\sqrt{6} & 1/\sqrt{3} & 0\\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2}\\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}\end{pmatrix}$$

Tri-bimaximal Mixing

Independent of mass eigenvalues !

Freedom of the rotation between 1st and 3rd column because a column corresponds to a mass eigenvalue.

Finally, we obtain PMNS matrix.

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6}+s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}-c/\sqrt{2} \\ -c/\sqrt{6}-s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}+c/\sqrt{2} \end{pmatrix}$$

$c = \cos\theta \quad s = \sin\theta e^{-i\sigma}$

CP violating phase appears accidentally.

Tri-maximal mixing : so called TM_2

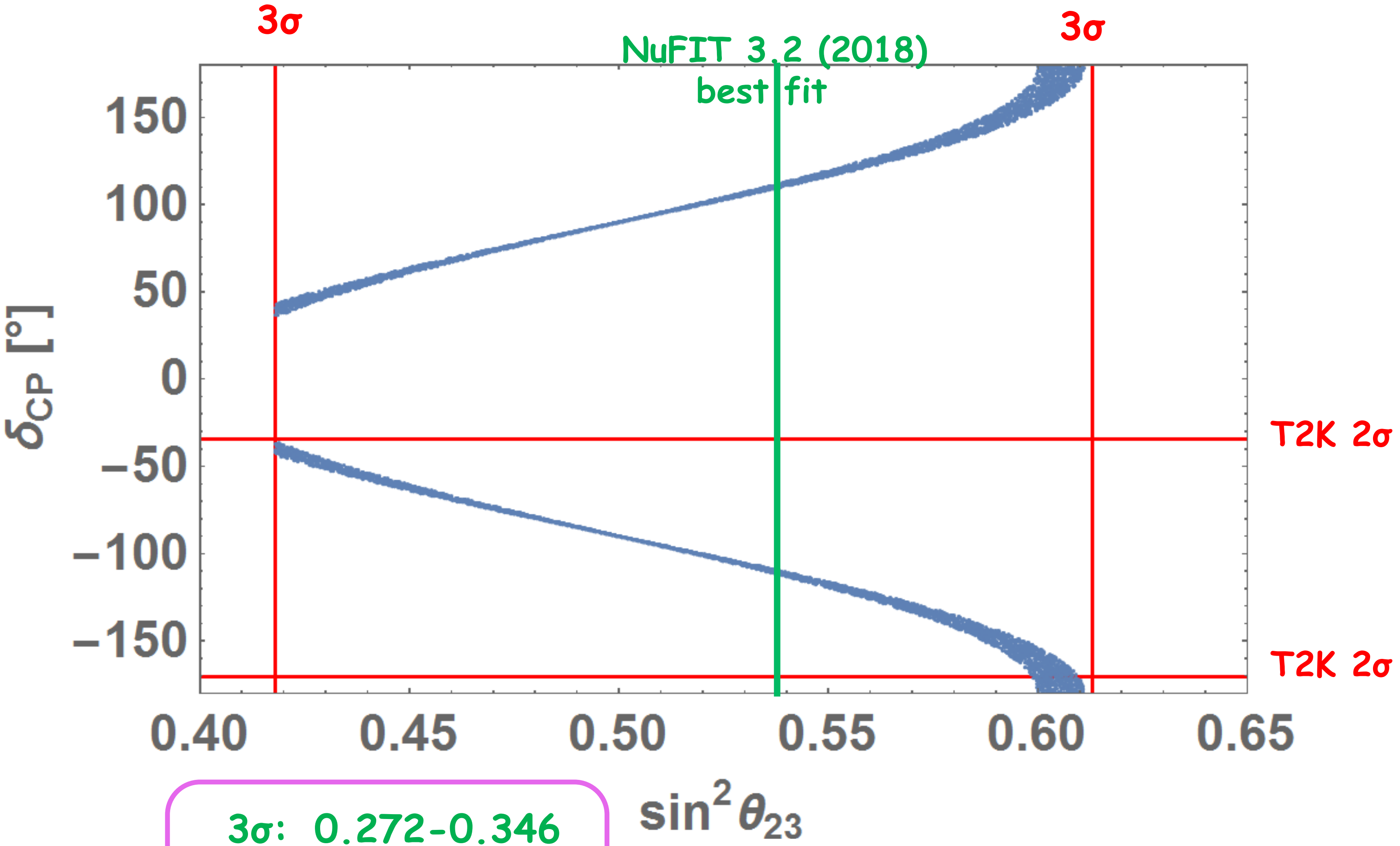
Θ and σ are not fixed.

Since two parameters appear, there are two relations among mixing angles and CP violating phase.

Mixing sum rules

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}\,, \qquad \cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1-\frac{5}{4}\sin^2\theta_{13}\right)$$

Prediction CP violating phase by using sum rules.



3σ: 0.272-0.346

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3} \ ,$$

$$\cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1 - \frac{5}{4}\sin^2\theta_{13}\right)$$

Direct Approach

☆ Flavor Structure of Yukawa Interactions is directly related with the Generators of Finite groups. Predictions are testable.

★ One cannot discuss the related phenomena without Lagrangian.
Leptogenesis, Quark CP violation, Lepton flavor violation

Model building is required.

☆ Conventional model building :
Introduce flavons (gauge singlet scalars) to discuss dynamics of flavors. Write down an effective Lagrangian including flavons. Flavor symmetry is broken spontaneously by VEV of flavons.

★ The number of parameters of Yukawa interactions increases.
Predictability of model is considerably reduced.

3 Prototype of Flavor model with A_4

Flavor symmetry G is broken by flavon (SU$_2$ singlet scalors) VEV's.
Flavor symmetry controls Yukaw couplings
among leptons and flavons with special vacuum alignments.

Consider the minimal number of flavons in A_4 model

	Leptons	flavons	
A_4 triplets	*L* (L_e, L_μ, L_τ)	$\phi_\nu(\phi_{\nu 1}, \phi_{\nu 2}, \phi_{\nu 3})$	couples to neutrino sector
		$\phi_E(\phi_{E1}, \phi_{E2}, \phi_{E3})$	couples to charged lepton sector
A_4 singlets	$e_R : 1 \quad \mu_R : 1" \quad \tau_R : 1'$		

Mass matrices are given by A_4 invariant Yukawa couplings with flavons

$$\mathbf{L} = y\, LL\, \Phi_\nu H_u H_u/\Lambda^2 + y_e\, Le^c\, \Phi_E H_d/\Lambda + y_\mu\, L\mu^c\, \Phi_E H_d/\Lambda + y_\tau\, L\tau^c\, \Phi_E H_d/\Lambda$$

$3_L \times 3_L \times 3_{\text{flavon}} \rightarrow 1$, $\qquad 3_L \times 1_R^{(')(")} \times 3_{\text{flavon}} \rightarrow 1$

Majoran neutrino

G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

Flavor symmetry G is broken by VEV of flavons

$\mathbf{3_L \times 3_L \times 3_{flavon} \rightarrow 1}$ $\mathbf{3_L \times 1_R (1_R', 1_R'') \times 3_{flavon} \rightarrow 1}$

$$m_{\nu LL} \sim y \begin{pmatrix} 2\langle\phi_{\nu1}\rangle & -\langle\phi_{\nu3}\rangle & -\langle\phi_{\nu2}\rangle \\ -\langle\phi_{\nu3}\rangle & 2\langle\phi_{\nu2}\rangle & -\langle\phi_{\nu1}\rangle \\ -\langle\phi_{\nu2}\rangle & -\langle\phi_{\nu1}\rangle & 2\langle\phi_{\nu3}\rangle \end{pmatrix} \qquad m_E \sim \begin{pmatrix} y_e\langle\phi_{E1}\rangle & y_e\langle\phi_{E3}\rangle & y_e\langle\phi_{E2}\rangle \\ y_\mu\langle\phi_{E2}\rangle & y_\mu\langle\phi_{E1}\rangle & y_\mu\langle\phi_{E3}\rangle \\ y_\tau\langle\phi_{E3}\rangle & y_\tau\langle\phi_{E2}\rangle & y_\tau\langle\phi_{E1}\rangle \end{pmatrix}$$

Residual symmetries lead to specific Vacuum Alingnments

Z_2 (1,S) in neutrinos $\langle\phi_{\nu1}\rangle = \langle\phi_{\nu2}\rangle = \langle\phi_{\nu3}\rangle$

Z_3 (1,T,T^2) in charged leptons $\langle\phi_{E2}\rangle = \langle\phi_{E3}\rangle = 0$

$$\Rightarrow \quad \langle\phi_\nu\rangle \sim (1,1,1)^T \ , \qquad \langle\phi_E\rangle \sim (1,0,0)^T$$

$$\boxed{S\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \ , \quad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}}$$

m_E is a diagonal matrix, on the other hand, $m_{\nu LL}$ is

$$m_{\nu LL} \sim 3y\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - y\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Rank 2

two generated masses and
one massless neutrinos !
(0, 3y, 3y)
Flavor mixing is not fixed !

Z_2 (1,S) is preserved

Adding A_4 singlet flavon $\xi : \mathbf{1}$ ➡ flavor mixing matrix is fixed.

G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

$$\mathbf{3_L} \times \mathbf{3_L} \times \mathbf{1_{flavon}} \rightarrow \mathbf{1}$$

$$m_{\nu LL} \sim y_1 \begin{pmatrix} 2\langle\phi_{\nu1}\rangle & -\langle\phi_{\nu3}\rangle & -\langle\phi_{\nu2}\rangle \\ -\langle\phi_{\nu3}\rangle & 2\langle\phi_{\nu2}\rangle & -\langle\phi_{\nu1}\rangle \\ -\langle\phi_{\nu2}\rangle & -\langle\phi_{\nu1}\rangle & 2\langle\phi_{\nu3}\rangle \end{pmatrix} + y_2\langle\xi\rangle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\langle\phi_{\nu1}\rangle = \langle\phi_{\nu2}\rangle = \langle\phi_{\nu3}\rangle$, which preserves S symmetry.

$$m_{\nu LL} = 3a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Flavor mixing is determined: Tri-bimaximal mixing. $\boxed{\theta_{13}=0}$

$$m_\nu = 3a + b,\ b,\ 3a - b \ \Rightarrow m_{\nu_1} - m_{\nu_3} = 2m_{\nu_2}$$

There appears a Neutrino Mass Sum Rule.

This is a minimal framework of A_4 symmetry predicting mixing angles and masses.

Prototype A_4 flavor model should be modified !

Need additional flavons in A_4 model

A_4 model realizes non-vanishing θ_{13} .

Y. Simizu, M. Tanimoto, A. Watanabe, PTP 126, 81(2011)

Add 1' or 1" flavon which couples to neutrinos.

$$\mathbf{LL}\quad \boxed{\mathbf{3}\times\mathbf{3}\Rightarrow\mathbf{1}} = a_1 * b_1 + a_2 * b_3 + a_3 * b_2$$

$$\mathbf{LL}\quad \mathbf{3}\times\mathbf{3}\Rightarrow\mathbf{1}' = a_1 * b_2 + a_2 * b_1 + a_3 * b_3$$

$$\mathbf{LL}\quad \boxed{\mathbf{3}\times\mathbf{3}\Rightarrow\mathbf{1}''} = a_1 * b_3 + a_2 * b_2 + a_3 * b_1$$

$$\overset{\boxed{\xi}}{\mathbf{1}\times\mathbf{1}\Rightarrow\mathbf{1}} \quad , \quad \overset{\boxed{\xi'}}{\mathbf{1}''\times\mathbf{1}'\Rightarrow\mathbf{1}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

Additional Matrix

$$M_\nu = a\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + b\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + c\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + d\boxed{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}$$

$$a = \frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{\Lambda}, \qquad b = -\frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{3\Lambda}, \qquad c = \frac{y^\nu_\xi \alpha_\xi v_u^2}{\Lambda}, \qquad d = \frac{y^\nu_{\xi'}\alpha_{\xi'} v_u^2}{\Lambda} \qquad a = -3b$$

$$M_\nu = V_{\text{tri-bi}}\begin{pmatrix}a+c-\frac{d}{2} & 0 & \frac{\sqrt{3}}{2}d\\ 0 & a+3b+c+d & 0\\ \frac{\sqrt{3}}{2}d & 0 & a-c+\frac{d}{2}\end{pmatrix}V^T_{\text{tri-bi}} \qquad \boxed{V_{\text{tri-bi}} = \begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{pmatrix}}$$

Predictions are consistent with the data of mixing angles for both normal and inverted mass hierarchies.

Predictability is reduced because of additional parameters.

3　Modular Group

Another aspect of A_4 model building

What is the origin of finite groups ?

It is well known that the superstring theory
on certain compactifications lead to
non-Abelian finite groups.

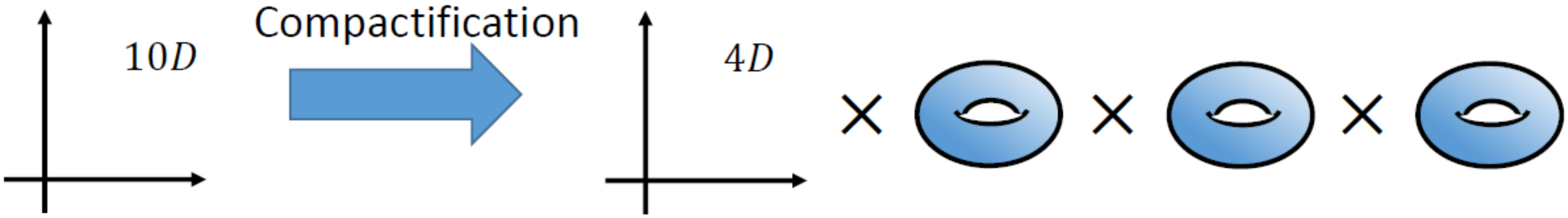
Indeed, torus compactification leads to Modular symmetery, which includes S_3, A_4, S_4, A_5 as its congruence subgroup.

R.Toorop, F.Feruglio, C.Hagedorn, arXiv:1112.1340;
F.Feruglio, arXiv:1706.08749; A_4　J.C.Criado, F.Feruglio, arXiv:1807.01125; A_4
J.T.Penedo, S.T.Petcov, arXiv:1806.11040; S_4
T.Kobayashi, K.Tanaka, T.H.Tatsuishi, arXiv:1803.10391; S_3
T.Kobayashi,N.Omoto,Y.Shimizu,K.Takagi,M.T,T.H.Tatsuishi, arXiv:1808.03012; A_4

Superstring theory 10D
Our universe is 4D

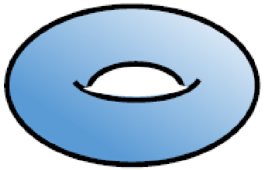
→ **The extra 6D should be compactified.**

Torus compactification



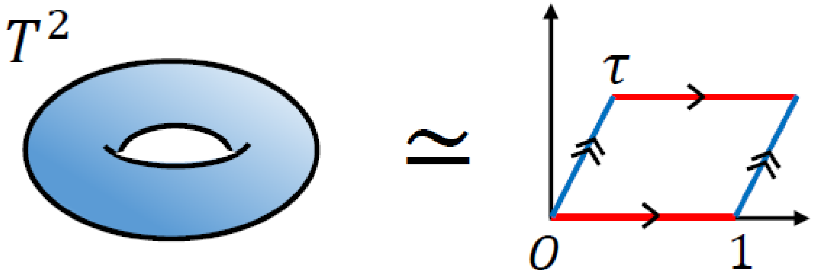
We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\mathrm{eff}}$$

→ $\mathcal{L}_{\mathrm{eff}}$ depends on the structure of 

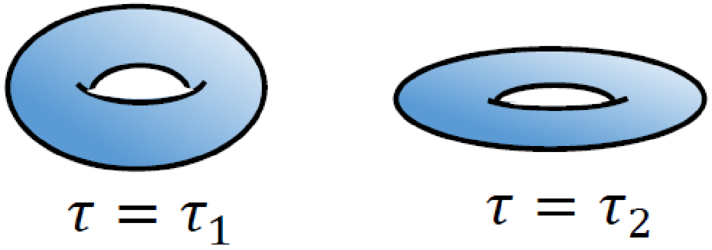
- $4D$ effective theory depends on internal space

$2D$ torus (T^2) is equivalent to parallelogram with identification of confronted sides.



Two-dimensional torus T^2 is obtained as $T^2 = \mathbb{R}^2 / \Lambda$
Λ is two-dimensional lattice

The shape of torus is represented by a modulus $\tau \in \mathbb{C}$.



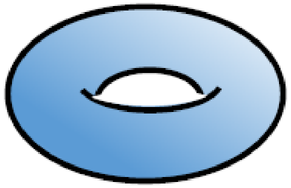
The different value of τ realize the different shape of T^2

$\mathcal{L}_{\text{eff}}$ depends on τ.

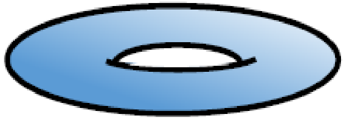
e.g.) $\mathcal{L}_{\text{eff}} \supset Y(\tau)_{ij} \phi \overline{\psi_i} \psi_j + \cdots$

- $4D$ effective theory depends on a modulus τ

The different value of τ realize the different shape of T^2



$\tau = \tau_1$

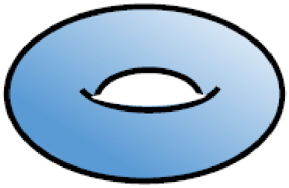


$\tau = \tau_2$

However,
there are specific transformations of τ which don't change T^2

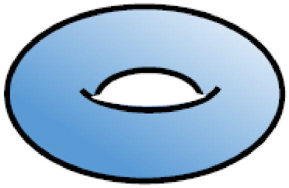
➡ Modular transformation

$\tau \to \tau'$



τ

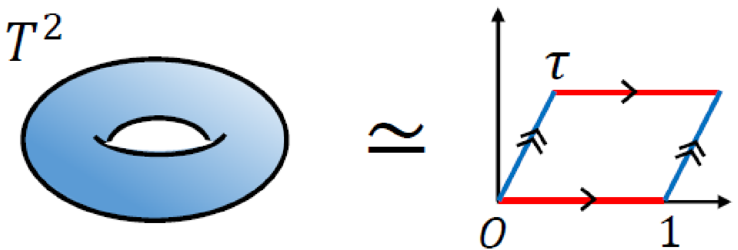
=

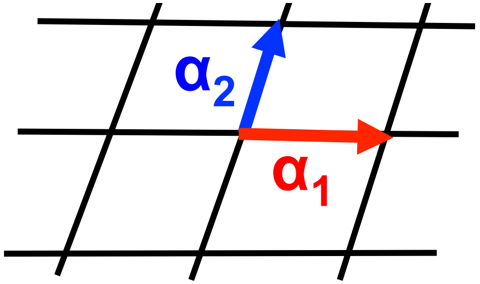


τ'

Modular transformation

The shape of a torus $T^2 \simeq$ The shape of a lattice on $\mathbb{C}$-plane





$(x,y) \sim (x,y) + n_1\alpha_1 + n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

**Λ is two-dimensional lattice,
which is spanned by two lattice vectors**

$$\alpha_1 = 2\pi R \quad \text{and} \quad \alpha_2 = 2\pi R \tau$$

$\tau = \alpha_2 / \alpha_1$ is a modulus parameter (complex).

The same lattice is spanned by other bases under the transformation

$$\boxed{\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}}$$

ad-bc=1
a,b,c,d are integer **SL(2,Z)**

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

$\tau = \alpha_2 / \alpha_1$

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

Modular transformation

Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

The modular transformation is generated by S and T .

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

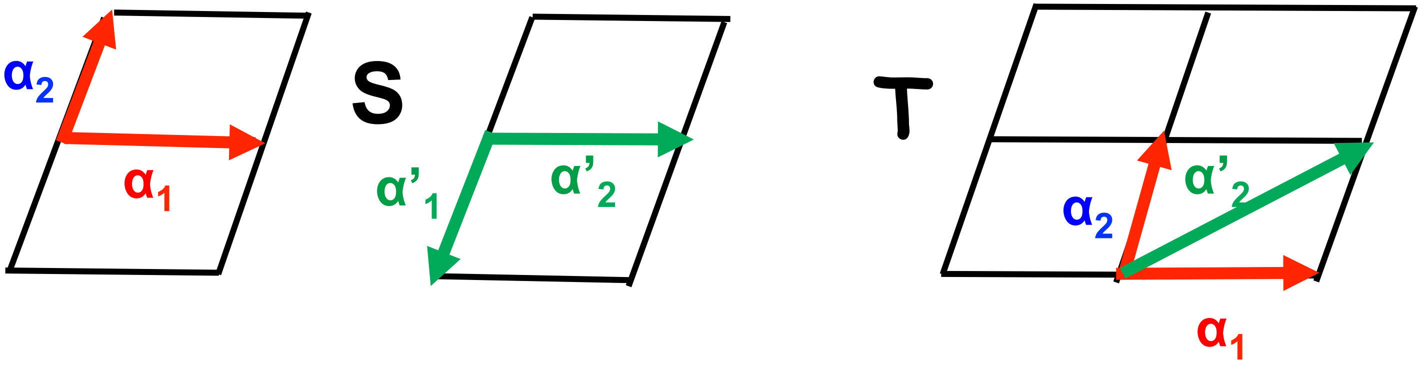
translation

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

$$T : \tau \longrightarrow \tau + 1$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$$



$$\tau = \alpha_2 / \alpha_1$$

$$S : \tau \longrightarrow -\frac{1}{\tau}, \qquad S^2 = 1, \qquad (ST)^3 = 1.$$
$$T : \tau \longrightarrow \tau + 1.$$

generate infinite discrete group

Modular group

4D effective theory

- **depends on a modulus τ**
- **is independent under modular transformation**

An example

$\mathcal{L}_1 = f(\tau)\phi_1\phi_2 \cdots \phi_n$

$f(\tau)$: coupling constant
ϕ_i: scalar fields

$f(\tau) \to (c\tau + d)^k f(\tau)$ ← Modular form with weight k

$\phi_i \to (c\tau + d)^{-k_i} \phi_i$

When $k = \sum_i k_i$, $\mathcal{L}_1$ is modular invariant.

Another example

$$\mathcal{L}_1 = f(\tau)\phi_1\phi_2\cdots\phi_n$$

- $f(\tau)$ and ϕ_i can be non-trivial representations of modular group Γ

Modular transformation:

SL(2, Z)

$$\gamma \in \Gamma \qquad \tau \to \frac{a\tau+b}{c\tau+d}, \qquad ad-bc=1$$

$$f(\tau) \to (c\tau+d)^{k}\rho(\gamma)f(\tau)$$

$$\phi_i' \to (c\tau+d)^{-k}\rho^{(i)}(\gamma)\phi_i$$

vanishing total modular weight

$\rho \times \rho^{I_1} \times \ldots \times \rho^{I_n}$ contains an invariant singlet

Representation matrix of Γ

L_1 is modular invariant.

Kinetic term is given by

$$\boxed{\frac{|\partial_\mu \phi_i|^2}{\langle \tau - \bar{\tau}\rangle^{k_i}}}$$

which is also invariant under modular transformation

- **Superpotential should be invariant under modular transformation in global SUSY model.**

Modular group has interesting subgroups

Modular group

$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$ **Infinite discrete group**

Impose $T^N=1$ congruence condition

$$\overline{\Gamma}(N) \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$$\Gamma(N) \equiv \Gamma / \overline{\Gamma}(N)$$

$\Gamma(2) \simeq S_3$, $\Gamma(3) \simeq A_4$, $\Gamma(4) \simeq S_4$, and $\Gamma(5) \simeq A_5$

We can consider effective theories with Γ(N) symmetry.

$$\mathcal{L}_{\text{eff}} \in \boxed{f(\tau)}\phi_1\phi_2\cdots\phi_n \qquad f(\tau), \phi_i\text{: non-trivial rep. of } \Gamma(N)$$

In some cases, explicit form of function f(τ) have been obtained.

Famous modular function : Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \boxed{\sqrt{-i\tau}\eta(\tau)}, \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

So called **Modular weight 1/2**

Modular transformation of chiral superfields in MSSM

$$\phi^{(I)} \to (c\tau+d)^{\boxed{-k_I}}\boxed{\rho^{(I)}(\gamma)}\phi^{(I)}$$

Modular weight

Representation matrix

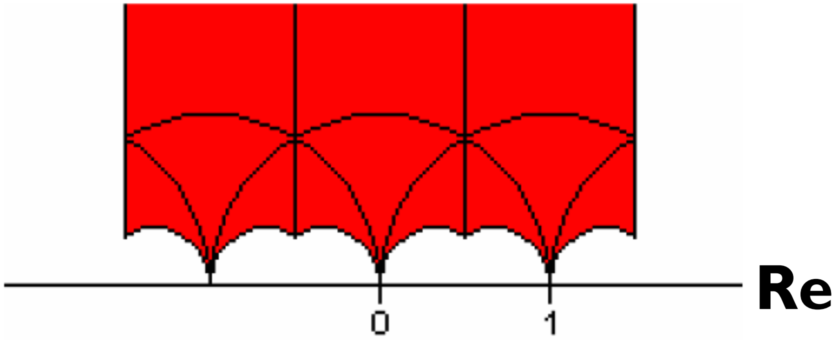
4 Predictions in Modular A_4 Symmetry

Take $T^3=1$

$\Gamma(3) \simeq$ **A_4 group**

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
③	0	$\boxed{2k+1}$	12	$\boxed{A_4}$
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

2k is weight



Fundamental domain of τ

There are 3 linealy independent modular forms for 2k=2 (weight 2)

Dimension $d_{2k}(\Gamma(3))=2k+1$

Triplet !

How to find A_4 triplet modular functions.

Prepare 4 Dedekind eta-functions as Modular functions

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$\begin{aligned}
\eta(3\tau) &\to \sqrt{\frac{-i\tau}{3}}\,\eta(\tau/3), \\
\eta(\tau/3) &\to \sqrt{-i3\tau}\,\eta(3\tau), \\
\eta((\tau+1)/3) &\to e^{-i\pi/12}\sqrt{-i\tau}\,\eta((\tau+2)/3), \\
\eta((\tau+2)/3) &\to e^{i\pi/12}\sqrt{-i\tau}\,\eta((\tau+1)/3).
\end{aligned}$$

S : $\tau \to -1/\tau$

$$\begin{aligned}
\eta(3\tau) &\to e^{i\pi/4}\eta(3\tau), \\
\eta(\tau/3) &\to \eta((\tau+1)/3), \\
\eta((\tau+1)/3) &\to \eta((\tau+2)/3), \\
\eta((\tau+2)/3) &\to e^{i\pi/12}\eta(\tau/3),
\end{aligned}$$

T : $\tau \to \tau+1$

Modular function with weight 2 by using Dedekind eta-function

$$Y(\alpha,\beta,\gamma,\delta|\tau)=\frac{d}{d\tau}(\alpha\log\eta(\tau/3)+\beta\log\eta((\tau+1)/3)+\gamma\log\eta((\tau+2)/3)+\delta\log\eta(3\tau))$$

$$\alpha+\beta+\gamma+\delta=0$$

$$S:\tau\longrightarrow-\frac{1}{\tau},\qquad T:\tau\longrightarrow\tau+1.$$

$$S:\quad Y(\alpha,\beta,\gamma,\delta|\tau)\to\tau^2Y(\delta,\gamma,\beta,\alpha|\tau),$$
$$T:\quad Y(\alpha,\beta,\gamma,\delta|\tau)\to Y(\gamma,\alpha,\beta,\delta|\tau).$$

In A_4 group, $T^3=1$

$$\rho(S)=\frac{1}{3}\begin{pmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{pmatrix},\qquad \rho(T)=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},$$

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular function with weight 2

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\ Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\ Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \end{aligned}$$

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$

$q = e^{2\pi i\tau}$

$|q| \ll 1$

$Y_2^2 + 2Y_1Y_3 = 0$

Simplest Model

left-handed leptons L(3) (L_e, L_μ, L_τ)
right-handed leptons e_R (1);μ_R (1”);τ_R (1’)

$-k_I$ is weight

	$SU(2)_L \times U(1)_Y$	A_4	k_I
$e^c_{R_1}$	$(1, +1)$	1	k_{e1}
$e^c_{R_2}$	$(1, +1)$	$1''$	k_{e2}
$e^c_{R_3}$	$(1, +1)$	$1'$	k_{e3}
L	$(2, -1/2)$	3	k_L
H_u	$(2, +1/2)$	1	k_{H_u}
H_d	$(2, -1/2)$	1	k_{H_d}
ϕ	$(1, 0)$	3	k_ϕ

Sum of weights should vanish

$-2k_L-2k_{Hu}\underset{Y}{+2} = 0$, $-k_L-k_{ei}-k_{Hd}\underset{Y}{+2} = 0$

Assign k_L=1, k_{ei}=1 k_{Hu}=k_{Hd}=0

Modular invariant superpotential

$$w_e = \alpha e_R H_d(LY) + \beta \mu_R H_d(LY) + \gamma \tau_R H_d(LY)$$

$$1_R^{(')('')} \times 3_L \times 3_Y \rightarrow 1$$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L Y)_1$$ Weinberg Operator

$$3_L \times 3_L \times 3_Y \rightarrow 1$$

$$M_E = \mathrm{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}$$

α, β, γ are fixed by the charged lepton masses

$$M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$$

Only source of breaking of the modular symmetry is the VEV of τ .

Unfortunately, the prediction is too large θ_{13} !

Seesaw model

Introduce right-handed neutrinos: A_4 Triplet

$$w_e = \alpha\ E_1^c H_d (L\ Y)_1 + \beta\ E_2^c H_d (L\ Y)_{1'} + \gamma\ E_3^c H_d (L\ Y)_{1''}$$

$$w_\nu = g(N^c H_u L\ Y)_1 + \Lambda (N^c N^c Y)_1$$

Sum of weights vanish.

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix} \qquad q = e^{2\pi i \tau}$$

M_E= $\alpha e_R H_d (LY) + \beta \mu_R H_d (LY) + \gamma \tau_R H_d (LY)$

A_4 1 1 3 3 1” 1 3 3 1’ 1 3 3

M_D= $g(\nu_R H_u L Y)_1$

A_4 3 1 3 3

M_N= $\Lambda(\nu_R \nu_R Y)_1$

A_4 3 3 3

seesaw $M_\nu = -M_D^T M_N^{-1} M_D$

$$
\overset{\nu_L}{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}}} \otimes \overset{\nu_R}{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}}
$$

$$
\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}}
$$

$$
\oplus \underbrace{\boxed{\frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}}}_{\text{symmetric} \times 3_Y} \oplus \underbrace{\boxed{\frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}}}_{\text{anti-symmetric} \times 3_Y} .
$$

Consider the case of Normal neutrino mass hierarchy

$m_1 < m_2 < m_3$

A_4 triplet 3 (Le, Lμ, Lτ) 3 (ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$)

A_4 singlets e_R 1 ; μ_R 1″ ; τ_R 1′

$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix}$$

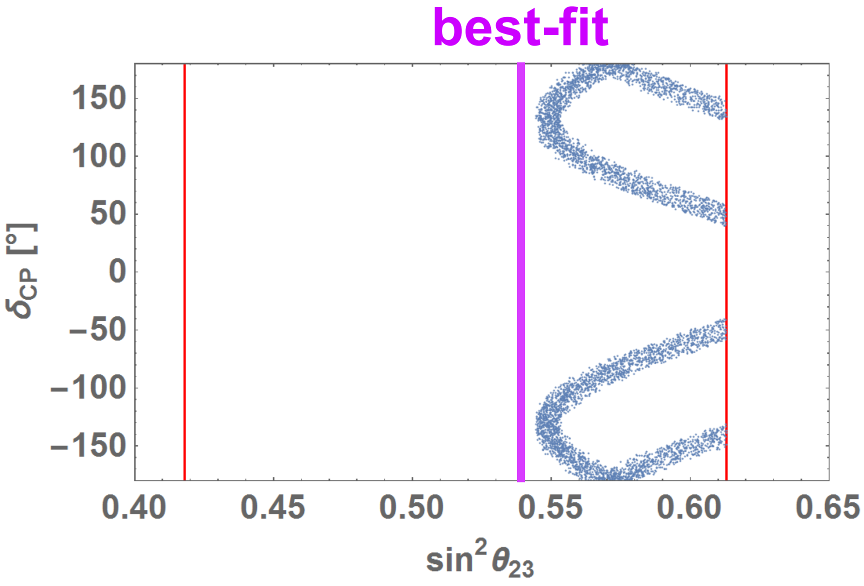
$$\mathcal{Y}_\nu = \begin{pmatrix} 2g_1 Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1 Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1 Y_3 \end{pmatrix}$$

$$\mathcal{M}_R = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \Lambda$$

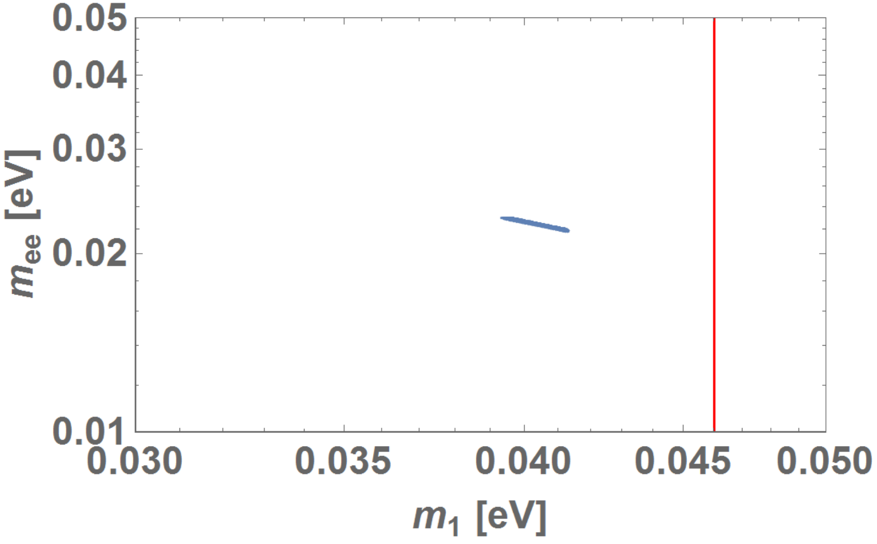
Parameters:
α, β, γ, $g_2/g_1=g$, τ

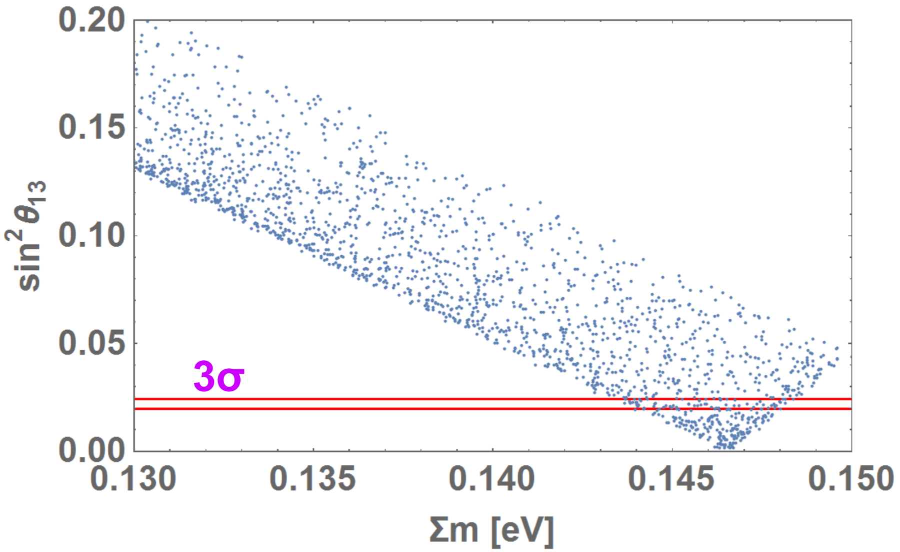
m_e, m_μ, m_τ fix α, β, γ .

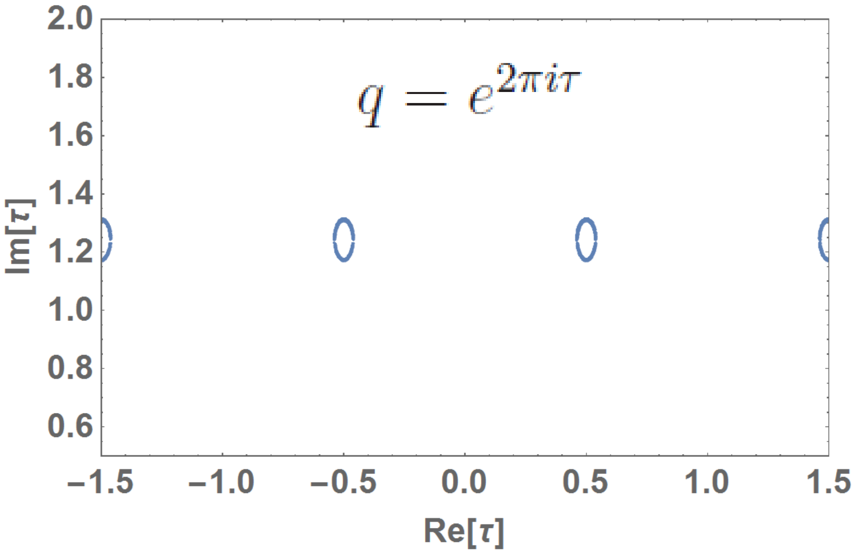
$\Delta m^2_{sol} / \Delta m^2_{atm}$ and θ_{23}, θ_{12}, θ_{13} fix $g_2/g_1=g$ and τ .



$m_1 \simeq m_2 \simeq 40$meV and $m_3 \simeq 60$meV

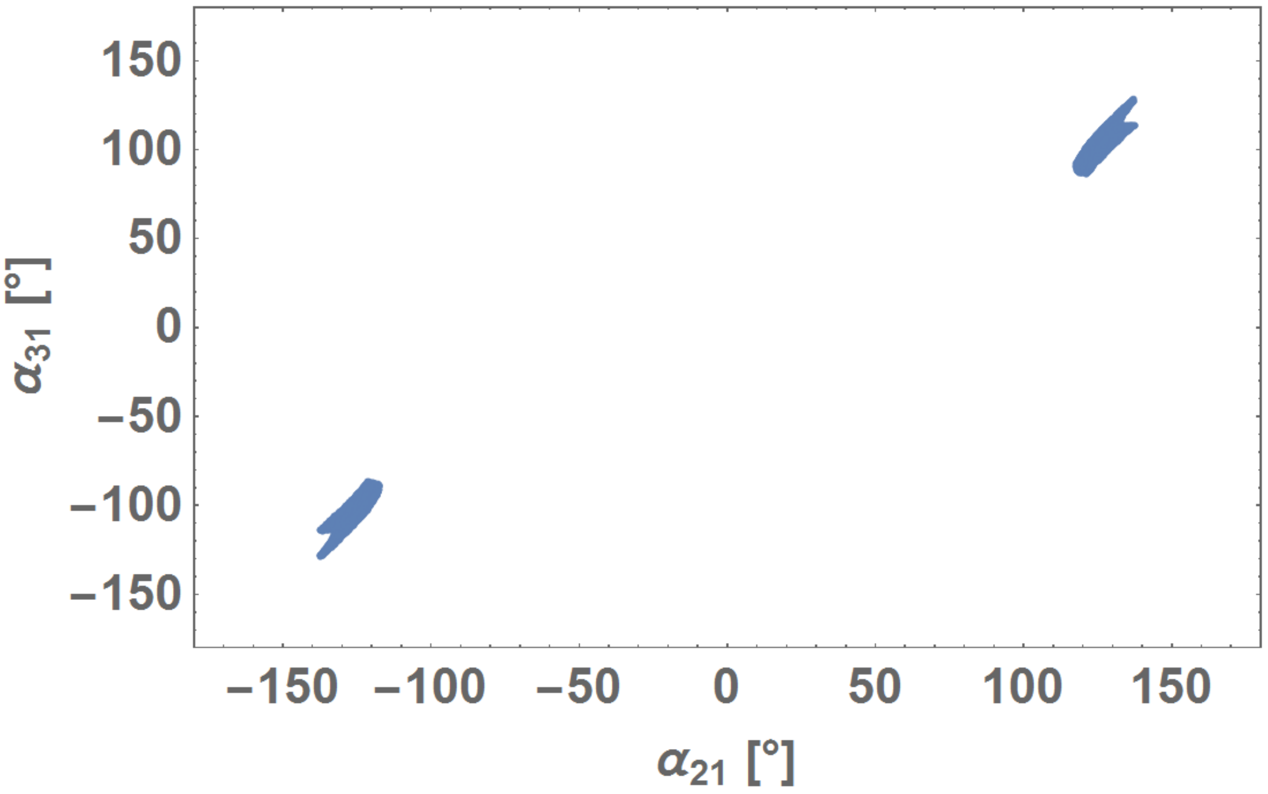




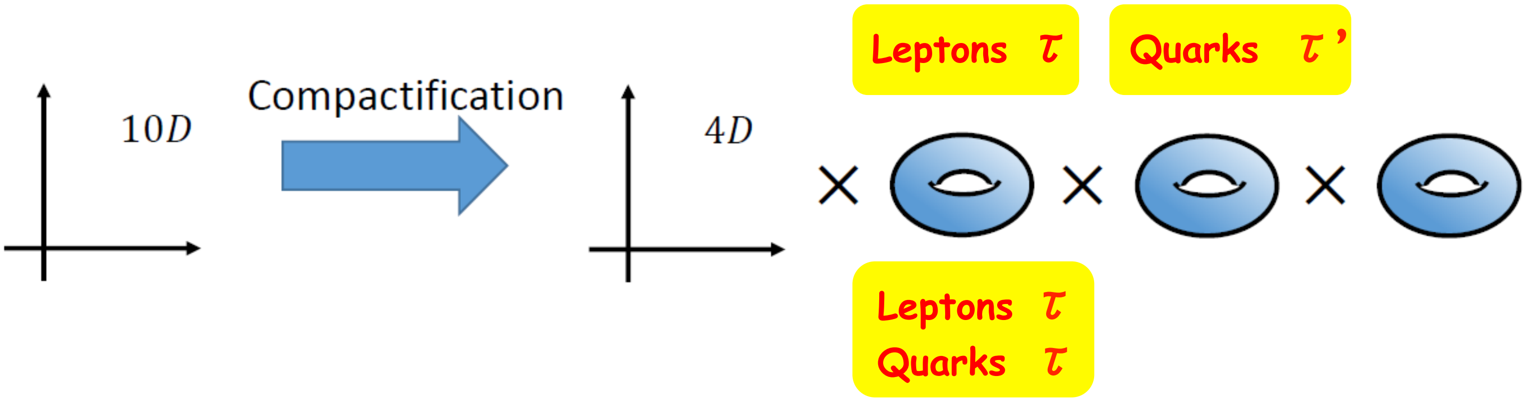


Arg [g] ∼ π/2

Predicted Majorana Phases



How is the quark mass matrix in modular A_4 symmetry ?



Simple model: left-handed doublet 3, right-handed singlet 1, 1”, 1’

$$\mathrm{diag}[\alpha,\beta,\gamma]\begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

for both up- and down-quarks

Coefficients α, β, γ are different for up- and down-quarks.

After fixing α, β, γ by inputting quark masses, one can examine CKM matrix elements by scanning modulus parameter τ .

6 Modular S_3 and S_4 Symmetries

$\Gamma(2) \simeq$ S_3 group

Irreducible representations: 1, 1', 2

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

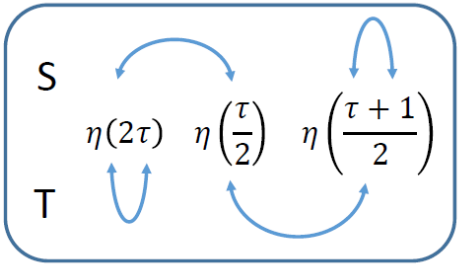
There are 2 linealy independent modular forms for weight 2 because of Dimension 2.

Doublet !

Prepare 3 Dedekind eta-functions as Modular functions

S

$$\eta(2\tau) \to \sqrt{\frac{-i\tau}{2}}\eta(\tau/2),$$
$$\eta(\tau/2) \to \sqrt{-i3\tau}\eta(2\tau),$$
$$\eta((\tau+1)/2) \to e^{-i\pi/12}\sqrt{-i\tau}\eta((\tau+1)/2).$$



T

$$\eta(2\tau) \to e^{i\pi/6}\eta(2\tau),$$
$$\eta(\tau/2) \to \eta((\tau+1)/2),$$
$$\eta((\tau+1)/2) \to e^{i\pi/12}\eta(\tau/2).$$

$$Y(\alpha,\beta,\gamma|\tau)=\frac{d}{d\tau}\left(\alpha\log\eta(\tau/2)+\beta\log\eta((\tau+1)/2)+\gamma\log\eta(2\tau)\right).$$

$$S:\quad Y(\alpha,\beta,\gamma|\tau)\to\tau^2Y(\gamma,\beta,\alpha|\tau),$$
$$T:\quad Y(\alpha,\beta,\gamma|\tau)\to Y(\gamma,\alpha,\beta|\tau).$$

$\alpha+\beta+\gamma=0$

$$\rho(S)=\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix},\qquad \rho(T)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

$$\begin{pmatrix}Y_1(-1/\tau)\\ Y_2(-1/\tau)\end{pmatrix}=\tau^2\rho(S)\begin{pmatrix}Y_1(\tau)\\ Y_2(\tau)\end{pmatrix},\qquad \begin{pmatrix}Y_1(\tau+1)\\ Y_2(\tau+1)\end{pmatrix}=\rho(T)\begin{pmatrix}Y_1(\tau)\\ Y_2(\tau)\end{pmatrix}.$$

$$\begin{aligned}Y_1(\tau)&=\frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)}+\frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}-\frac{8\eta'(2\tau)}{\eta(2\tau)}\right),\\ Y_2(\tau)&=\frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)}-\frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right),\end{aligned}$$

$$\begin{aligned}Y_1(\tau)&=\frac{1}{8}+3q+3q^2+12q^3+3q^4\cdots,\\ Y_2(\tau)&=\sqrt{3}q^{1/2}(1+4q+6q^2+8q^3\cdots).\end{aligned}$$

The model is consistent with the experimental data for only inverted mass hierarchy.

$\Gamma(4) \simeq S_4$ group

Irreducible representations: 1, 1’, 2, 3, 3’

J. Penedo, S. Petcov arXiv:1806.11040

There are 5 linealy independent modular forms for weight 2 because of Dimension 5.

Doublet + Triplet !

Prepare 6 Dedekind eta-functions as Modular functions

$$S: \begin{cases} \eta\left(\tau+\frac{1}{2}\right) & \to \frac{1}{\sqrt{2}}\sqrt{-i\tau}\,\eta\left(\frac{\tau+2}{4}\right) \\ \eta\left(4\tau\right) & \to \frac{1}{2}\sqrt{-i\tau}\,\eta\left(\frac{\tau}{4}\right) \\ \eta\left(\frac{\tau}{4}\right) & \to 2\sqrt{-i\tau}\,\eta\left(4\tau\right) \\ \eta\left(\frac{\tau+1}{4}\right) & \to e^{-i\pi/6}\sqrt{-i\tau}\,\eta\left(\frac{\tau+3}{4}\right) \\ \eta\left(\frac{\tau+2}{4}\right) & \to \sqrt{2}\sqrt{-i\tau}\,\eta\left(\tau+\frac{1}{2}\right) \\ \eta\left(\frac{\tau+3}{4}\right) & \to e^{i\pi/6}\sqrt{-i\tau}\,\eta\left(\frac{\tau+1}{4}\right) \end{cases}$$

$$T: \begin{cases} \eta\left(\tau+\frac{1}{2}\right) & \to e^{i\pi/12}\,\eta\left(\tau+\frac{1}{2}\right) \\ \eta\left(4\tau\right) & \to e^{i\pi/3}\,\eta\left(4\tau\right) \\ \eta\left(\frac{\tau}{4}\right) & \to \eta\left(\frac{\tau+1}{4}\right) \\ \eta\left(\frac{\tau+1}{4}\right) & \to \eta\left(\frac{\tau+2}{4}\right) \\ \eta\left(\frac{\tau+2}{4}\right) & \to \eta\left(\frac{\tau+3}{4}\right) \\ \eta\left(\frac{\tau+3}{4}\right) & \to e^{i\pi/12}\,\eta\left(\frac{\tau}{4}\right) \end{cases} .$$

$$Y(a_1,\ldots,a_6|\tau) \equiv \frac{d}{d\tau}\left(\sum_{i=1}^{6} a_i \,\log \eta_i(\tau)\right)$$

$\Sigma\ a_i = 0$

$$= a_1 \frac{\eta'(\tau+1/2)}{\eta(\tau+1/2)} + 4\,a_2 \frac{\eta'(4\tau)}{\eta(4\tau)} + \frac{1}{4}\Bigg[a_3 \frac{\eta'(\tau/4)}{\eta(\tau/4)} + a_4 \frac{\eta'((\tau+1)/4)}{\eta((\tau+1)/4)} + a_5 \frac{\eta'((\tau+2)/4)}{\eta((\tau+2)/4)} + a_6 \frac{\eta'((\tau+3)/4)}{\eta((\tau+3)/4)}\Bigg]$$

$$S:\ Y(a_1,\ldots,a_6|\tau)\ \to\ Y(a_1,a_2,a_3,a_4,a_5,a_6|-1/\tau)\ =\ \tau^2\,Y(a_5,a_3,a_2,a_6,a_1,a_4|\tau)\,,$$

$$T:\ Y(a_1,\ldots,a_6|\tau)\ \to\ Y(a_1,a_2,a_3,a_4,a_5,a_6|\tau+1)\ =\ Y(a_1,a_2,a_6,a_3,a_4,a_5|\tau)\,.$$

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}.$$

$$\begin{pmatrix} Y_3(-1/\tau) \\ Y_4(-1/\tau) \\ Y_5(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_3(\tau+1) \\ Y_4(\tau+1) \\ Y_5(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}.$$

$$S^2\ =\ (ST)^3\ =\ T^4\ =\ \mathbb{1}$$

$$\mathbf{2}: \quad \rho(S) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{3}: \quad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$$\mathbf{3}': \quad \rho(S) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$$\begin{aligned}
Y_1(\tau) &\equiv Y(1,1,\omega,\omega^2,\omega,\omega^2|\tau), \\
Y_2(\tau) &\equiv Y(1,1,\omega^2,\omega,\omega^2,\omega|\tau), \\
Y_3(\tau) &\equiv Y(1,-1,-1,-1,1,1|\tau), \\
Y_4(\tau) &\equiv Y(1,-1,-\omega^2,-\omega,\omega^2,\omega|\tau), \\
Y_5(\tau) &\equiv Y(1,-1,-\omega,-\omega^2,\omega,\omega^2|\tau),
\end{aligned}$$

$$Y_{\mathbf{2}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}$$

$$Y_{\mathbf{3}'}(\tau) \equiv \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$

3'

No solution of weight 2 for 3

Seesaw model is consistent with the experimental data of mixing.

However, phenomenological implications are not discussed enough.

7 Summary

- Footprint of the non-Abelian discrete symmetry is expected to be seen in the neutrino mixing matrix. It is an imprint of generators of finite groups. A_4 …..
- A_4 is a congruence subgroup of the modular group, which comes from superstring theory on certain compactifications.
- Mass matrices of A_4 model are determined essentially by the modular parameter τ .
- Predictions are sharp and testable in the future.
- Is Modulus τ common in both quarks and leptons ?
- S_3 and S_4 are also subgroups of the modular group.

We need more phenomenological discussions.